GELFAND-TSETLIN BASIS FOR THE REPRESENTATIONS OF $\mathfrak{gl}_n$

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1. Finite dimensional representations of $\mathfrak{gl}_n$

Let e_{ij}, $i, j = 1, \ldots, n$ denote the standard basis of the general linear Lie algebra $\mathfrak{gl}_n$ over the field of complex numbers. The subalgebra $\mathfrak{gl}_{n-1}$ is spanned by the basis elements e_{ij} with $i, j = 1, \ldots, n-1$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{gl}_n$. The elements $e_{11}, \ldots, e_{nn}$ form a basis of $\mathfrak{h}$.

Finite-dimensional irreducible representations of $\mathfrak{gl}_n$ are in a one-to-one correspondence with n-tuples of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}^+ \quad \text{for } i = 1, \ldots, n-1. \tag{1.1}$$

Such an n-tuple λ is called the highest weight of the corresponding representation which we shall denote by $L(\lambda)$. It contains unique, up to a multiple, nonzero vector v^+ (the highest vector) such that $e_{ii}v^+ = \lambda_i v^+$ for $i = 1, \ldots, n$ and $e_{ij}v^+ = 0$ for $1 \leqslant i < j \leqslant n$.

2. BGG resolution

Last seminar, we talked about the BGG resolution. Now let us recall the BGG resolution and deduce the Weyl character formula from it.

Denote by M_λ the Verma module with highest weight λ. Let W be the Weyl group and s_i be the simple refections, $i = 1, \ldots, \text{rank}\, \mathfrak{g}$ and ρ is the Weyl vector. Define the shifted action of the Weyl group on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. Denote P^+ and Φ^+ the set of all dominant integral weights and the set of all positive roots of $\mathfrak{g}$.

Theorem 2.1 ([BGG]). *Let $\lambda \in \mathcal{P}^+$. Then there exists a lon exact sequence*

$$0 \to M_{w_0 \cdot \lambda} \to \cdots \bigoplus_{w \in W, l(w) = k} M_{w \cdot \lambda} \cdots \to \bigoplus_i M_{s_i \cdot \lambda} \to M_\lambda \to L(\lambda) \to 0 \tag{2.1}$$

where $l(w)$ is the length of an element $w \in W$, and w_0 is the longest element of the Weyl group.

Now we are going to deduce the Weyl character formula from Theorem 2.1.

Note that the character of M_λ is

$$\text{ch}(M_\lambda) = q^\lambda \prod_{\alpha \in \Phi^+} \frac{1}{1 - q^{-\alpha}} = q^\lambda \prod_{\alpha \in \Phi^+} (1 + q^{-\alpha} + q^{-2\alpha} + \cdots).$$

Since character is additive, we can take the alternating sum with respect to character in (2.1). It follows that

$$\text{ch}(L(\lambda)) = \sum_{w \in W} (-1)^w \text{ch}(M_{w \cdot \lambda}) = \frac{\sum_{w \in W} (-1)^w q^{w \cdot \lambda}}{\prod_{\alpha \in \Phi^+} (1 - q^{-\alpha})}. \tag{2.2}$$

Let $\lambda = 0$, we have

$$\prod_{\alpha \in \Phi^+} (1 - q^{-\alpha}) = \sum_{w \in W} (-1)^w q^{w\rho - \rho}.$$

It follows that

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^w q^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^w q^{w\rho}}, \tag{2.3}$$

which is the celebrated Weyl character formula.

Since we are interested in the case $\mathfrak{g} = \mathfrak{gl}_n$. We know that W is the symmetric group $\mathcal{S}_n$. Also, we have $\rho = (n-1,\ldots,1,0)$. If we write x_i for $q^{(0,\ldots,1,\ldots,0)}$, where $(0,\ldots,1,\ldots,0)$ has all components zero except for the i-th component. Then we can rewrite (2.3) as

$$\mathrm{ch}(L(\lambda)) = \frac{\det\left(x_i^{\lambda_i+n-j}\right)_{1\leqslant i,j\leqslant n}}{\det\left(x_i^{n-j}\right)_{1\leqslant i,j\leqslant n}} = S_\lambda(x_1,\ldots,x_n),$$

where S_λ is exactly the Schur polynomial.

Our next goal is to deduce the Branching rule for the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ from Schur polynomial. First we need the following proposition

Proposition 2.2. *We have*

$$S_\lambda(x_1,\ldots,x_{n-1},x_n=1) = \sum_\mu S_\mu(x_1,\ldots,x_{n-1}),$$

where μ runs over all partitions such that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n.$$

Proof. We prove it only for the case $n=3$. And from this we can definitely get the idea to prove it for general n. Let $\lambda = (\lambda_1,\lambda_2,\lambda_3)$, $\lambda_i \in \mathbb{Z}^+$. We have

$$S_\lambda(x_1,x_2,x_3) = \frac{1}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}\begin{vmatrix} x_1^{\lambda_1+2} & x_1^{\lambda_2+1} & x_1^{\lambda_3} \\ x_2^{\lambda_1+2} & x_2^{\lambda_2+1} & x_2^{\lambda_3} \\ x_3^{\lambda_1+2} & x_3^{\lambda_2+1} & x_3^{\lambda_3} \end{vmatrix}.$$

Setting $x_3 = 1$, we have

$$S_\lambda(x_1,x_2,1) = \frac{1}{(x_1-x_2)(x_1-1)(x_2-1)}\begin{vmatrix} x_1^{\lambda_1+2} & x_1^{\lambda_2+1} & x_1^{\lambda_3} \\ x_2^{\lambda_1+2} & x_2^{\lambda_2+1} & x_2^{\lambda_3} \\ 1 & 1 & 1 \end{vmatrix}.$$

Subtracting the 2nd column from the 1st, and then 3rd column from the 2nd, we get

$$S_\lambda(x_1,x_2,1) = \frac{1}{(x_1-x_2)(x_1-1)(x_2-1)}\begin{vmatrix} x_1^{\lambda_1+2} - x_1^{\lambda_2+1} & x_1^{\lambda_2+1} - x_1^{\lambda_3+1} \\ x_2^{\lambda_1+2} - x_2^{\lambda_2+1} & x_2^{\lambda_2+1} - x_2^{\lambda_3+1} \end{vmatrix}.$$

We see that the 1st row can be divided by $x_1 - 1$, and the second one by $x_2 - 1$.

Note that for $a, b \in \mathbb{Z}^+$ such that $a > b$:

$$\frac{x^a - x^b}{x-1} = \sum_{c:b\leqslant c<a} x^a.$$

We see that by dividing by $x_1 - 1$ in the first row in the first place we get a sum which can be written as

$$\sum_{\lambda_2+1\leqslant\mu_1+1<\lambda_1+2} x_1^{\mu+1} = \sum_{\lambda_2\leqslant\mu_1\leqslant\lambda_1} x_1^{\mu+1}.$$

We choose to call the summation index $\mu_1 + 1$ so that we will have the desired result. Considering three other matrix elements in a similar way, we conclude the proof. □

Corollary 2.3. *We have*

$$S_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_\mu S_\mu(x_1, \dots, x_{n-1}) x_n^{|\lambda|-|\mu|},$$

where μ runs over all partitions such that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \dots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n.$$

From Proposition 2.2, one can easily get the branching rule for the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$.

Theorem 2.4. *The restriction of $L(\lambda)$ to the subalgebra $\mathfrak{gl}_{n-1}$ is isomorphic to the direct sum of pairwise inequivalent irreducible representations*

$$L(\lambda)\big|_{\mathfrak{gl}_{n-1}} \cong \bigoplus_\mu L'_\mu$$

summed over the highest weights μ satisfying the betweenness conditions

$$\lambda_i - \mu_i \in \mathbb{Z}^+ \quad \text{and} \quad \mu_i - \lambda_{i+1} \in \mathbb{Z}^+ \quad \text{for } i = 1, \dots, n-1. \tag{2.4}$$

3. Gelfand-Tsetlin basis

The next two sections are copied from [M].

The subsequent applications of the branching rule to the subalgebras of the chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

yield a parameterization of basis vectors in $L(\lambda)$ by the combinatorial objects called the Gelfand-Tsetlin patterns. Such a pattern Λ (associate with λ) is an array of row vectors

$$\begin{array}{ccccccccc} \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{n,n-1} & & \lambda_{nn} \\ & \lambda_{n-1,1} & & & \cdots & & & \lambda_{n-1,n-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{21} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array}$$

where the upper row coincides with λ and the following conditions hold

$$\lambda_{ki} - \lambda_{k-1,i} \in \mathbb{Z}^+, \quad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}^+, \quad i = 1, \dots, k-1$$

for each $k = 2, \dots, n$.

The Gelfand-Tsetlin basis of $L(\lambda)$ is provided by the following theorem. Let us set $l_{ki} = \lambda_{ki} - i + 1$.

Theorem 3.1 ([GT]). *There exists a basis $\{v_\Lambda\}$ in $L(\lambda)$ parametrized by all patterns Λ such that the action of generators of $\mathfrak{gl}_n$ is given by the formulas*

$$e_{kk} v_\Lambda = \Big(\sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \Big) v_\Lambda, \tag{3.1}$$

$$e_{k,k+1} v_\Lambda = -\sum_{i=1}^{k} \frac{(l_{ki} - l_{k+1,1}) \cdots (l_{ki} - l_{k+1,k+1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})} v_{\Lambda + \delta_{ki}}, \tag{3.2}$$

$$e_{k+1,k}v_\Lambda = \sum_{i=1}^{k} \frac{(l_{ki}-l_{k-1,1})\cdots(l_{ki}-l_{k-1,k-1})}{(l_{ki}-l_{k1})\cdots\wedge\cdots(l_{ki}-l_{kk})} v_{\Lambda-\delta_{ki}}. \tag{3.3}$$

The arrays $\Lambda \pm \delta_{ki}$ are obtained from Λ by replacing λ_{ki} by $\lambda_{ki} \pm 1$. It is supposed that $v_\Lambda = 0$ if the array Λ is not a pattern; the symbol $\wedge$ indicates that the zero factor in the denominator is skipped.

The Gelfand-Tsetlin subalgebra of $\mathcal{U}(\mathfrak{gl}_n)$ is the subalgebra $\mathcal{H}_n$ generated by the centers $\mathcal{Z}(\mathfrak{gl}_1), \cdots, \mathcal{Z}(\mathfrak{gl}_n)$ of the subalgebra chain

$$\mathcal{U}(\mathfrak{gl}_1) \subset \mathcal{U}(\mathfrak{gl}_2) \subset \cdots \subset \mathcal{U}(\mathfrak{gl}_n) \quad \text{induced by} \quad \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n.$$

Theorem 3.2 ([V]). *The Gelfand-Tsetlin subalgebra $\mathcal{H}_n$ of $\mathcal{U}(\mathfrak{gl}_n)$ is maximal commutative.*

4. Construction of the basis: lowering and raising operators

For each $i = 1, \ldots, n-1$ introduce the following elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$

$$z_{in} = \sum_{i>i_1>\cdots>i_s\geqslant 1} e_{ii_1}e_{i_1i_2}\ldots e_{i_{s-1}i_s}e_{i_sn}(h_i-h_{j_1})\ldots(h_i-h_{j_r}), \tag{4.1}$$

$$z_{ni} = \sum_{i<i_1<\cdots<i_s\leqslant n} e_{i_1i}e_{i_2i_1}\ldots e_{i_si_{s-1}}e_{ni_s}(h_i-h_{j_1})\ldots(h_i-h_{j_r}), \tag{4.2}$$

where s runs over nonnegative integers, $h_i = e_{ii} - i + 1$ and $\{j_1, \ldots, j_r\}$ is the complementary subset to $\{i_1, \ldots, i_s\}$ in the set $\{1, \ldots, i-1\}$ or $\{i+1, \ldots, n-1\}$, respectively. For instance,

$$z_{13} = e_{13}, \quad z_{23} = e_{23}(e_{22}-e_{11}-1) + e_{21}e_{13},$$
$$z_{32} = e_{32}, \quad z_{31} = e_{31}(e_{11}-e_{22}+1) + e_{21}e_{32}.$$

Consider now the irreducible finite-dimensional representation $L(\lambda)$ of $\mathfrak{gl}_n$ with the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the highest vector v^+. Denote by $L(\lambda)^+$ the subspace of $\mathfrak{gl}_{n-1}$-singular vectors in $L(\lambda)$:

$$L(\lambda)^+ = \{v \in L(\lambda) \mid e_{ij}v = 0, \quad 1 \leqslant i < j \leqslant n\}.$$

Given a $\mathfrak{gl}_{n-1}$-weight $\mu = (\mu_1, \ldots, \mu_{n-1})$ we denote by $L(\lambda)^+_\mu$ the corresponding weight subspace in $L(\lambda)^+$:

$$L(\lambda)^+_\mu = \{v \in L(\lambda)^+ \mid e_{ii}v = \mu_i v, \quad i = 1, \ldots, n-1\}.$$

The main property of the elements z_{ni} and z_{in} is described by the following lemma.

Lemma 4.1. *Let $v \in L(\lambda)^+_\mu$. Then for any $i = 1, \ldots, n-1$ we have*

$$z_{in}v \in L(\lambda)^+_{\mu+\delta_i} \quad \text{and} \quad z_{ni}v \in L(\lambda)^+_{\mu-\delta_i},$$

where the weight $\mu \pm \delta_i$ is obtained from μ by replacing μ_i with $\mu_i \pm 1$.

This result allows us to regard the elements z_{in} and z_{ni} as operators in the space $L(\lambda)^+$. They are called the raising and lowering operators, respectively. By the branching rule (Theorem 2.4) the space $L(\lambda)^+_\mu$ is one-dimensional if the conditions (2.4) hold and it is zero otherwise.

Lemma 4.2. *Suppose that μ satisfies the betweenness conditions (2.4). Then the vector*

$$v_\mu = z_{n1}^{\lambda_1-\mu_1}\cdots z_{n,n-1}^{\lambda_{n-1}-\mu_{n-1}}v^+$$

is nonzero. Moreover, the space $L(\lambda)^+_\mu$ is spanned by v_μ.

The $\mathcal{U}(\mathfrak{gl}_{n-1})$-span of each nonzero vector v_μ is a $\mathfrak{gl}_{n-1}$-module isomorphic to $L'(\mu)$. Iterating the construction of the vectors v_μ for each pair of Lie algebras $\mathfrak{gl}_{k-1}\subset\mathfrak{gl}_k$ we shall be able to get a basis in the entire space $\Lambda(\lambda)$.

Theorem 4.3. *The basis vector v_Λ of Theorem 3.1 can be given by the formula*

$$v_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow}\left(z_{k1}^{\lambda_{k,1}-\lambda_{k-1,1}}\cdots z_{k,k-1}^{\lambda_{k,k-1}-\lambda_{k-1,k-1}}\right)v^+, \tag{4.3}$$

where the factors in the product are ordered in accordance with increase of the indices.

Instead of giving a proof first, I will check in Section 5 that the construction in Theorem 4.3 does satisfy the formulas in Theorem 3.1 for some simple cases. The proof will be given in Section 6.

5. Examples

In this section, we are going to check that the vectors constructed from Theorem 4.3 does satisfy the formulas in Theorem 3.1 for $n=2,3$. The case for $\mathfrak{gl}_3$ is not complete.

5.1. **The case of $\mathfrak{gl}_2$.** Let us fix the highest weight $\lambda=(\lambda_1,\lambda_2)$, then $\lambda_{21}=\lambda_1$, $\lambda_{22}=\lambda_2$. Denote by $\mu=\lambda_{11}$, then each pattern is determined by $k=\lambda_1-\mu$ for $k=0,\dots,\lambda_1-\lambda_2$. By (4.3), the vector corresponds to k is given by $v_k=e_{21}^{\lambda_1-\mu}v^+=e_{21}^kv^+$. Now it is trivial that $e_{21}v_k=v_{k+1}$, which is exactly (3.3)(note that $\mu-1$ corresponds to $k+1$).

Since for any vector v of weight (μ_1,μ_2), the vector $e_{21}v$ has weight (μ_1-1,μ_2+1). Hence $v_k=e_{21}^kv^+$ has weight $(\lambda_1-k,\lambda_2+k)$. Therefore, we have

$$e_{11}v_k=(\lambda_1-k)v_k=\mu v_k=\lambda_{11}v_k,$$

$$e_{22}v_k=(\lambda_2+k)v_k=(\lambda_{21}+\lambda_{22}-\lambda_{11})v_k.$$

Formula (3.1) is checked.

Finally, to check (3.2), we need the following useful formula in $\mathcal{U}(gl_2)$

$$[e_{12},e_{21}^k]=-ke_{21}^{k-1}(k-1-e_{11}+e_{22}). \tag{5.1}$$

Now (3.2) holds because of

$$\begin{aligned}
e_{12}v_k =& e_{12}e_{21}^kv^+ = e_{21}^ke_{12}v^+ + [e_{12},e_{21}^k]v^+\\
=&-ke_{21}^{k-1}(k-1-e_{11}+e_{22})v^+\\
=&-(\lambda_1-\mu)(\lambda_1-\mu-1-\lambda_1+\lambda_2)v_{k-1}\\
=&-(\lambda_{11}-\lambda_{21})(\lambda_{11}-\lambda_{22}+1)v_{k-1}\\
=&-(l_{11}-l_{21})(l_{11}-l_{22})v_{k-1}.
\end{aligned}$$

5.2. **The case of $\mathfrak{gl}_3$.** This case is more involved. First we fix the highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (\lambda_{31}, \lambda_{32}, \lambda_{33})$. Then each pattern is unique determined by $\lambda_{21}, \lambda_{22}$ and λ_{11}. We denote the vector constructed in Theorem 4.3 corresponding to this patter by $(\lambda_{21}, \lambda_{22}; \lambda_{11})$, then

$$(\lambda_{21}, \lambda_{22}; \lambda_{11}) = e_{21}^{\lambda_{21}-\lambda_{11}}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+.$$

It is easy to see

$$\begin{aligned} e_{21}(\lambda_{21}, \lambda_{22}; \lambda_{11}) =& e_{21}^{\lambda_{21}-\lambda_{11}+1}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ \\ =&(\lambda_{21}, \lambda_{22}; \lambda_{11}-1). \end{aligned}$$

Since for any vector v of weight (μ_1, μ_2, μ_3), the vectors $e_{21}v$, $e_{32}v$ and $e_{31}v$ have weight $(\mu_1 - 1, \mu_2 + 1, \mu_3)$, $(\mu_1, \mu_2 - 1, \mu_3 + 1)$ and $(\mu_1 - 1, \mu_2, \mu_3 + 1)$ respectively. Moreover, $e_{11} - e_{22} + 1$ acts on v by scalar product. One has that the weight of $(\lambda_{21}, \lambda_{22}; \lambda_{11})$ is

$$(\lambda_{11}, \lambda_{21}+\lambda_{22}-\lambda_{11}, \lambda_{31}+\lambda_{32}+\lambda_{33}-\lambda_{21}-\lambda_{22}).$$

Hence we get formulas (3.1)

$$\begin{aligned} e_{11}(\lambda_{21}, \lambda_{22}; \lambda_{11}) &= \lambda_{11}(\lambda_{21}, \lambda_{22}; \lambda_{11}), \\ e_{22}(\lambda_{21}, \lambda_{22}; \lambda_{11}) &= (\lambda_{21}+\lambda_{22}-\lambda_{11})(\lambda_{21}, \lambda_{22}; \lambda_{11}), \\ e_{33}(\lambda_{21}, \lambda_{22}; \lambda_{11}) &= (\lambda_{31}+\lambda_{32}+\lambda_{33}-\lambda_{21}-\lambda_{22})(\lambda_{21}, \lambda_{22}; \lambda_{11}). \end{aligned}$$

Now let us consider $e_{12}(\lambda_{21}, \lambda_{22}; \lambda_{11})$. By (5.1), we have

$$\begin{aligned} & e_{12}(\lambda_{21}, \lambda_{22}; \lambda_{11}) \\ =& e_{12}e_{21}^{\lambda_{21}-\lambda_{11}}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ \\ =& e_{21}^{\lambda_{21}-\lambda_{11}}e_{12}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ - (\lambda_{21}-\lambda_{11})e_{21}^{\lambda_{21}-\lambda_{11}-1} \\ & \times (\lambda_{22}-\lambda_{11}-1-e_{11}+e_{22})(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ \\ =& e_{21}^{\lambda_{21}-\lambda_{11}}e_{12}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ \\ & -(\lambda_{21}-\lambda_{11})(\lambda_{22}-\lambda_{11}-1)(\lambda_{21}, \lambda_{22}; \lambda_{11}+1). \end{aligned}$$

For the last equality, we use the fact that $(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+$ has weight $(\lambda_{21}, \lambda_{22}, \lambda_{31}+\lambda_{32}+\lambda_{33}-\lambda_{21}-\lambda_{22})$. Now it is enough to check that

$$e_{21}^{\lambda_{21}-\lambda_{11}}e_{12}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})^{\lambda_{31}-\lambda_{21}}e_{32}^{\lambda_{32}-\lambda_{22}}v^+ = 0. \tag{5.2}$$

We compute $e_{12}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32})$ first. We have

$$\begin{aligned} & e_{12}(e_{31}(e_{11}-e_{22}+1)+e_{21}e_{32}) \\ =& (e_{31}e_{12}-e_{32})(e_{11}-e_{22}+1)+e_{21}e_{12}e_{32}+(e_{11}-e_{22})e_{32} \\ =& e_{31}e_{12}(e_{11}-e_{22})+e_{31}e_{12}-e_{32}(e_{11}-e_{22})-e_{32}+e_{21}e_{32}e_{12}+(e_{11}-e_{22})e_{32} \\ =& e_{31}(e_{11}-e_{22})e_{12}-2e_{31}e_{12}+e_{31}e_{12}+e_{32}-e_{32}+e_{21}e_{32}e_{12} \\ =& (e_{31}(e_{11}-e_{22}-1)+e_{21}e_{32})e_{12}. \end{aligned}$$

Now one has (5.2) since v^+ is a singular vector and $[e_{12}, e_{32}] = 0$.

6. Proof of Theorem 3.1

It suffices to show that with the formulas given by (3.1), (3.2) and (3.3) the vector space $\bigoplus_\Lambda v_\Lambda$ dose give us a simple representation of $\mathfrak{gl}_n$ with highest weight λ.

First we need to check it gives us a representation. Let A be the Cartan matrix of $\mathfrak{sl}_n$, and set $e_i = e_{i,i+1}$, $f_i = e_{i+1,i}$ and $h_i = e_{ii} - e_{i+1,i+1}$ for every $i = 1, \dots, n-1$. Then the Serre relations for $\mathfrak{sl}_n$ are

$$[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i,$$

$$[h_i, h_j] = 0, \quad \mathrm{Ad}_{e_i}^{1-a_{ij}} e_j = \mathrm{Ad}_{f_i}^{1-a_{ij}} f_j = 0,$$

for every $i, j = 1, \dots, n-1$. Hence it is enough to check that

$$[h_i, e_j]v_\Lambda = a_{ij}e_jv_\Lambda, \quad [h_i, f_j]v_\Lambda = -a_{ij}f_jv_\Lambda, \quad [e_i, f_j]v_\Lambda = \delta_{ij}h_iv_\Lambda,$$

$$[h_i, h_j]v_\Lambda = 0, \quad \mathrm{Ad}_{e_i}^{1-a_{ij}} e_jv_\Lambda = \mathrm{Ad}_{f_i}^{1-a_{ij}} f_jv_\Lambda = 0,$$

for every $i, j = 1, \dots, n-1$ and Gelfand-Tsetlin pattern Λ. To our convenience, for each given pattern Λ associated to λ, we write $\Lambda^{(k)} := \sum_{i=1}^k \lambda_{ki}$, and

$$E_{ki}^\Lambda := \frac{(l_{ki} - l_{k+1,1}) \cdots (l_{ki} - l_{k+1,k+1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})}, \quad F_{ki}^\Lambda := \frac{(l_{ki} - l_{k-1,1}) \cdots (l_{ki} - l_{k-1,k-1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})} \tag{6.1}$$

for every $k = 1, \dots, n$ and $i = 1, \dots, k$.

Because h_i and h_j diagonally act on $\bigoplus_\Lambda v_\Lambda$, hence $[h_i, h_j]v_\Lambda = 0$. To prove $[h_i, e_j]v_\Lambda = a_{ij}e_jv_\Lambda$, it suffices to show

$$[e_{ii}, e_{j,j+1}]v_\Lambda = \delta_{i,k}e_{i,j+1}v_\Lambda - \delta_{j+1,i}e_{ji}v_\Lambda. \tag{6.2}$$

Since the patterns involved in $e_{j,j+1}v_\Lambda$ are given by increasing one of the entries in the j-th row of Λ by 1 and the action of e_{ii} on v_Λ only involves the sum of i-th and $(i-1)$-th row of Λ, one can easily check (6.2). Hence we have $[h_i, e_j]v_\Lambda = a_{ij}e_jv_\Lambda$ and similarly $[h_i, f_j]v_\Lambda = -a_{ij}f_jv_\Lambda$.

Now let us prove $[e_i, f_j]v_\Lambda = \delta_{ij}h_iv_\Lambda$. With the notation in (6.1), we have

$$e_{i,i+1}e_{j+1,j}v_\Lambda = e_{i,i+1}\sum_{k=1}^{j} F_{jk}^\Lambda v_{\Lambda-\delta_{jk}} = -\sum_{k=1}^{j}\sum_{s=1}^{i} F_{jk}^\Lambda E_{is}^{\Lambda-\delta_{jk}} v_{\Lambda-\delta_{jk}+\delta_{is}},$$

$$e_{j+1,j}e_{i,i+1}v_\Lambda = -e_{j,j+1}\sum_{s=1}^{i} E_{is}^\Lambda v_{\Lambda+\delta_{is}} = -\sum_{k=1}^{j}\sum_{s=1}^{i} F_{jk}^{\Lambda+\delta_{is}} E_{is}^\Lambda v_{\Lambda-\delta_{jk}+\delta_{is}}.$$

Since F_{jk}^Λ only involves the j-th and $(j-1)$-th rows of Λ while E_{is}^Λ only involves the i-th and $(i+1)$-th rows, we have

$$F_{jk}^\Lambda E_{is}^{\Lambda-\delta_{jk}} = F_{jk}^{\Lambda+\delta_{is}} E_{is}^\Lambda \tag{6.3}$$

for $i \neq j$ and $i \neq j-1$. If $i = j-1$, then we have

$$\frac{F_{jk}^\Lambda}{F_{jk}^{\Lambda+\delta_{is}}} = \frac{l_{jk} - l_{j-1,s}}{l_{jk} - l_{j-1,s} - 1}, \quad \frac{E_{is}^\Lambda}{E_{is}^{\Lambda-\delta_{jk}}} = \frac{l_{is} - l_{i+1,k}}{l_{is} - l_{i+1,k} + 1} = \frac{l_{jk} - l_{j-1,s}}{l_{jk} - l_{j-1,s} - 1},$$

i.e. (6.3) holds. If $i=j$ and $s\neq k$, one can check (6.3) directly. Therefore $[e_i,f_j]v_\Lambda=\delta_{ij}h_iv_\Lambda$ is equivalent to

$$\sum_{k=1}^{i}\Big(\frac{(l_{ik}+1-l_{i-1,1})\dots(l_{ik}+1-l_{i-1,i-1})}{(l_{ik}+1-l_{i1})\cdots\wedge\dots(l_{ik}+1-l_{ii})}\cdot\frac{(l_{ik}-l_{i+1,1})\dots(l_{ik}-l_{i+1,i+1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})}$$
$$-\frac{(l_{ik}-l_{i-1,1})\dots(l_{ik}-l_{i-1,i-1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})}\cdot\frac{(l_{ik}-1-l_{i+1,1})\dots(l_{ik}-1-l_{i+1,i+1})}{(l_{ik}-1-l_{i1})\cdots\wedge\dots(l_{ik}-1-l_{ii})}\Big)$$
$$=2(l_{i1}+\dots+l_{ii})-(l_{i-1,1}+\dots+l_{i-1,i-1})-(l_{i+1,1}+\dots+l_{i+1,i+1})-1. \tag{6.4}$$

Denote the right hand side of (6.4) by $\mathcal{M}$, then $\mathcal{M}$ is a symmetric rational function with respect to $l_{i1},\dots,l_{ii}$. Moreover, it has at most simple poles. Note that a symmetric function cannot have simple poles, it follows that $\mathcal{M}$ is a polynomial in $l_{i1},\dots,l_{ii}$. By considering the asymptotic behavior, we know that $\mathcal{M}\sim 2l_{i1}$ as l_{i1} tends to infinity. By symmetry, it follows that $\mathcal{M}-2(l_{i1}+\dots+l_{ii})$ is independent of $l_{i1},\dots,l_{ii}$.

It is easy to see $l_{i+1,1}$ is a linear term in $\mathcal{M}$. Let us consider the coefficient of $l_{i+1,1}$. This coefficient is

$$-\sum_{k=1}^{i}\Big(\frac{(l_{ik}+1-l_{i-1,1})\dots(l_{ik}+1-l_{i-1,i-1})}{(l_{ik}+1-l_{i1})\cdots\wedge\dots(l_{ik}+1-l_{ii})}\cdot\frac{(l_{ik}-l_{i+1,2})\dots(l_{ik}-l_{i+1,i+1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})}$$
$$-\frac{(l_{ik}-l_{i-1,1})\dots(l_{ik}-l_{i-1,i-1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})}\cdot\frac{(l_{ik}-1-l_{i+1,2})\dots(l_{ik}-1-l_{i+1,i+1})}{(l_{ik}-1-l_{i1})\cdots\wedge\dots(l_{ik}-1-l_{ii})}\Big).$$

Let $\varphi(x)=(x+1-l_{i-1,1})\dots(x+1-l_{i-1,i-1})(x-l_{i+1,2})\dots(x-l_{i+1,i+1})$, then $\deg\varphi=2i-1$. Set $a_k=l_{ik}$ and $a_{k+i}=l_{ik}-1$ for every $k=1,\dots,i$, we have

$$\varphi(a_k)\prod_{j\neq k}\frac{1}{(a_k-a_j)}=\frac{(l_{ik}+1-l_{i-1,1})\dots(l_{ik}+1-l_{i-1,i-1})}{(l_{ik}+1-l_{i1})\cdots\wedge\dots(l_{ik}+1-l_{ii})}\times\frac{(l_{ik}-l_{i+1,2})\dots(l_{ik}-l_{i+1,i+1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})},$$

$$\varphi(a_{k+i})\prod_{j\neq k+i}\frac{1}{(a_k-a_j)}=-\frac{(l_{ik}-l_{i-1,1})\dots(l_{ik}-l_{i-1,i-1})}{(l_{ik}-l_{i1})\cdots\wedge\dots(l_{ik}-l_{ii})}\times\frac{(l_{ik}-1-l_{i+1,2})\dots(l_{ik}-1-l_{i+1,i+1})}{(l_{ik}-1-l_{i1})\cdots\wedge\dots(l_{ik}-1-l_{ii})}.$$

By Lagrange Interpolating Polynomial, we have

$$\sum_{k=1}^{2i}\varphi(a_k)\prod_{j\neq k}\frac{(x-a_j)}{(a_k-a_j)}=\varphi(x).$$

Consider the leading coefficient, it follows that

$$\sum_{k=1}^{2i}\varphi(a_k)\prod_{j\neq k}\frac{1}{(a_k-a_j)}=1.$$

Hence the coefficient of $l_{i+1,1}$ is equal to -1. By symmetry all the coefficient of $l_{i+1,k}$ are equal to -1 for every $k=1,\dots,i+1$. Similarly, all the coefficient of $l_{i-1,k}$ are equal to -1

for every $k = 1, \dots, i-1$. Now it is easy to see

$$\mathcal{M} - \big(2(l_{i1} + \cdots + l_{ii}) - (l_{i-1,1} + \cdots + l_{i-1,i-1}) - (l_{i+1,1} + \cdots + l_{i+1,i+1})\big)$$

is a constant. Let $l_{sk} = 2k$ for every $s = i-1, i, i+1$ and $k = 1, \dots, s$, we can get this constant, which is exactly -1. Hence we proved $[e_i, f_j]v_\Lambda = \delta_{ij} h_i v_\Lambda$.

Remark 6.1. *A simple proof of* (6.4)*, as given by Prof. Tarasov, is the following. Let* $p_j(u) = (u - l_{j1}) \dots (u - l_{jj})$*, then the left hand side of* (6.4) *is equal to*

$$-\sum_{k=1}^{i} \mathrm{Res}_{u=l_{ki}} \frac{p_{i-1}(u) p_{i+1}(u-1)}{p_i(u) p_i(u-1)} - \sum_{k=1}^{i} \mathrm{Res}_{u=l_{ki}+1} \frac{p_{i-1}(u) p_{i+1}(u-1)}{p_i(u) p_i(u-1)}$$

and the right hand side of (6.4) *is equal to*

$$\mathrm{Res}_{u=\infty} \frac{p_{i-1}(u) p_{i+1}(u-1)}{p_i(u) p_i(u-1)}.$$

Hence they must equal. □

Similar to the first half of the proof of $[e_i, f_j]v_\Lambda = \delta_{ij} h_i v_\Lambda$, we can prove $[e_{i,i+1}, e_{j,j+1}]v_\Lambda = 0$ if $|i-j| \neq 0$ ($i = j$ is trivial). Let us consider $[e_{i,i+1}, [e_{i,i+1}, e_{i+1,i+2}]]v_\Lambda$. First, we have $[x,[x,y]] = x^2 y + y x^2 - 2xyx$ and

$$e_{i,i+1}^2 e_{i+1,i+2} v_\Lambda = -\sum_{k=1}^{i+1} \sum_{s=1}^{i} \sum_{t=1}^{i} E_{i+1,k}^{\Lambda} E_{is}^{\Lambda+\delta_{i+1,k}} E_{it}^{\Lambda+\delta_{i+1,k}+\delta_{is}} v_{\Lambda+\delta_{i+1,k}+\delta_{is}+\delta_{it}},$$

$$2 e_{i,i+1} e_{i+1,i+2} e_{i,i+1} v_\Lambda = -2 \sum_{k=1}^{i+1} \sum_{s=1}^{i} \sum_{t=1}^{i} E_{i+1,k}^{\Lambda+\delta_{is}} E_{is}^{\Lambda} E_{it}^{\Lambda+\delta_{i+1,k}+\delta_{is}} v_{\Lambda+\delta_{i+1,k}+\delta_{is}+\delta_{it}},$$

$$e_{i+1,i+2} e_{i,i+1}^2 v_\Lambda = -\sum_{k=1}^{i+1} \sum_{s=1}^{i} \sum_{t=1}^{i} E_{i+1,k}^{\Lambda+\delta_{is}+\delta_{it}} E_{is}^{\Lambda} E_{it}^{\Lambda+\delta_{is}} v_{\Lambda+\delta_{i+1,k}+\delta_{is}+\delta_{it}}.$$

To prove $[e_{i,i+1}, [e_{i,i+1}, e_{i+1,i+2}]]v_\Lambda = 0$, it suffices to show the coefficient of $v_{\Lambda+\delta_{i+1,k}+\delta_{is}+\delta_{it}}$ is zero. If $s \neq t$, after taking the common factors, the coefficient is a multiple of

$$\begin{aligned} &\frac{(l_{is} - l_{i+1,k} - 1)(l_{it} - l_{i+1,k} - 1)}{(l_{is} - l_{it})(l_{it} - l_{is} - 1)} - 2\frac{(l_{is} - l_{i+1,k})(l_{it-l_{i+1,k}} - 1)}{(l_{is} - l_{it})(l_{it} - l_{is} - 1)} + \frac{(l_{is} - l_{i+1,k})(l_{it} - l_{i+1,k})}{(l_{is} - l_{it})(l_{it} - l_{is} - 1)} \\ +&\frac{(l_{it} - l_{i+1,k} - 1)(l_{is} - l_{i+1,k} - 1)}{(l_{it} - l_{is})(l_{is} - l_{it} - 1)} - 2\frac{(l_{it} - l_{i+1,k})(l_{is-l_{i+1,k}} - 1)}{(l_{it} - l_{is})(l_{is} - l_{it} - 1)} + \frac{(l_{it} - l_{i+1,k})(l_{is} - l_{i+1,k})}{(l_{it} - l_{is})(l_{is} - l_{it} - 1)}. \end{aligned}$$

It is easy to see the first line of this expression is equal to $\dfrac{1}{l_{is} - l_{it}}$ and the second $\dfrac{1}{l_{it} - l_{is}}$. Hence the coefficient is zero. Similarly, if $s = t$, then it suffices to check

$$(l_{is} - l_{i+1,k} - 1)(l_{is} - l_{i+1,k}) - 2(l_{is} - l_{i+1,k})(l_{is} - l_{i+1,k}) + (l_{is} - l_{i+1,k})(l_{is} + 1 - l_{i+1,k}) = 0,$$

which is obvious. Similarly, we can prove $\mathrm{Ad}_{e_i}^{1-a_{ij}} e_j v_\Lambda = \mathrm{Ad}_{f_i}^{1-a_{ij}} f_j v_\Lambda = 0$.

Let Λ_0 be the Gelfand-Testlin pattern associated to λ defined by $\lambda_{ki} = \lambda_i$ for every $k = 1, \dots, n$ and $i = 1, \dots, k$. Now by the formulas (3.1) and (3.2), we have

$$e_{ii} v_{L_0} = \lambda_i v_{\Lambda_0}, \quad e_{i,i+1} v_{\Lambda_0} = 0.$$

Hence v_{Λ_0} is a singular vector with weight λ. Note that by definition, $\bigoplus_\Lambda v_\Lambda$ is finite-dimensional. Moreover, by Branching rule (Theorem 2.4), the dimension of $L(\lambda)$ is equal to the number of Gelfand-Testlin patterns associated to λ. It follows that $L(\lambda) = \bigoplus_\Lambda v_\Lambda$, and the proof of Theorem 3.1 is complete.

7. Further topics

We might consider how the Gelfand-Tsetlin algebra acts on this basis. To be continued.

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