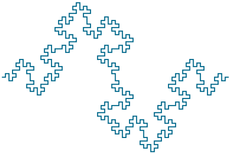
Bethe Ansatz method in Gaudin Model

Via examples of the vector representation of $\mathfrak{gl}_{r+1}$

Kang Lu



April 22, 2016

Lie algebra

- Lie algebra $\mathfrak{gl}_{r+1}$

Bethe Ansatz in Gaudin model

- Gaudin model
- Bethe Ansatz
- Weight function

Completeness

- Obatained results for completeness of types B,C,D

Connections and proposed research

- Selberg integral
- Orthogonal polynomial
- Fuchsian differential operators
- Schubert calculus

Lie algebra
- Lie algebra $\mathfrak{gl}_{r+1}$

Bethe Ansatz in Gaudin model
- Gaudin model
- Bethe Ansatz
- Weight function

Completeness
- Obatained results for completeness of types B,C,D

Connections and proposed research
- Selberg integral
- Orthogonal polynomial
- Fuchsian differential operators
- Schubert calculus

LIE ALGEBRA $\mathfrak{gl}_{r+1}$

Let $E_{i,j}$ for $i = 1, \ldots, r+1$ be the standard generators of $\mathfrak{gl}_{r+1}$. The commutator relations are given by

$$[E_{i,j}, E_{s,k}] = \delta_{j,s} E_{i,k} - \delta_{i,k} E_{s,j}.$$

We have the triangular decomposition

$$\mathfrak{gl}_{r+1} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where

$$\mathfrak{n}_+ = \bigoplus_{i<j} \mathbb{C} \cdot E_{i,j}, \quad \mathfrak{h} = \bigoplus_{i=1}^{r+1} \mathbb{C} \cdot E_{i,i}, \quad \mathfrak{n}_- = \bigoplus_{i>j} \mathbb{C} \cdot E_{i,j}.$$

$\mathfrak{sl}_{r+1}$ is the simple Lie algebra of type A with Lie algebra rank r. The representations of $\mathfrak{sl}_{r+1}$ are almost the same as those of $\mathfrak{gl}_{r+1}$.

VECTOR REPRESENTATION

Set $V = \mathbb{C}^{r+1}$. Let $e_i \in \mathbb{C}^{r+1}$ be the vector with 1 on the i-th component and 0 on all the other components, then $\{e_i\}_{i=1}^{r+1}$ is a basis of V. Define the standard action of $\mathfrak{gl}_{r+1}$ on V by

$$E_{i,j}e_k = \delta_{j,k}e_i.$$

V is the **vector representation** of $\mathfrak{gl}_{r+1}$. Let $v_+ = e_1$ be the highest weight vector of V, we have

$$E_{i,i}v_+ = \delta_{1,i}v_+, \quad \mathfrak{n}_+ v_+ = 0, \quad \text{for } i = 1, \ldots, r+1.$$

V is the irreducible $\mathfrak{gl}_{r+1}$ representation with highest weight ω_1, where $\omega_1 \in \mathfrak{h}^*$ and $\omega_1(E_{i,i}) = \delta_{1,i}$.

Tensor product of Vector representations

Let n be a positive integer. If $X \in \mathrm{End}(V)$, then we denote by $X^{(i)} \in \mathrm{End}(V^{\otimes n})$ the operator $\mathrm{id}^{\otimes i-1} \otimes X \otimes \mathrm{id}^{\otimes n-i}$ acting non-trivially on the i-th factor of the tensor product. Then for any $X \in \mathfrak{gl}_{r+1}$, the action of X on $V^{\otimes n}$ is given by $\sum_{i=1}^{n} X^{(i)}$.

Example: $E_{11}(v_+ \otimes v_+ \otimes \cdots \otimes v_+)$
$= E_{11}v_+ \otimes v_+ \otimes \cdots \otimes v_+ + v_+ \otimes E_{11}v_+ \otimes \cdots \otimes v_+ + \cdots + v_+ \otimes v_+ \otimes \cdots \otimes E_{11}v_+$
$= n(v_+ \otimes v_+ \otimes \cdots \otimes v_+)$.

Let $(V^{\otimes n})^{\mathrm{sing}} = \{v \in V^{\otimes n} \mid \mathfrak{n}_+ v = 0\}$ be the subspace of **singular vectors** in $V^{\otimes n}$. For $\mu \in \mathfrak{h}^*$ let

$$(V^{\otimes n})_\mu = \{v \in V^{\otimes n} \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}^*\}$$

be the subspace of $V^{\otimes n}$ of vectors of **weight** μ. Denote the **singular space of weight** μ in $V^{\otimes n}$ by $(V^{\otimes n})_\mu^{\mathrm{sing}} = (V^{\otimes n})^{\mathrm{sing}} \cap (V^{\otimes n})_\mu$.

Lie algebra
- Lie algebra $\mathfrak{gl}_{r+1}$

Bethe Ansatz in Gaudin model
- Gaudin model
- Bethe Ansatz
- Weight function

Completeness
- Obatained results for completeness of types B,C,D

Connections and proposed research
- Selberg integral
- Orthogonal polynomial
- Fuchsian differential operators
- Schubert calculus

Gaudin model

The **Gaudin model** was introduced by [Gaudin 1976] for the case of Lie algebra $\mathfrak{gl}_2$ and later generalized by [Gaudin 1983] for all simple Lie algebras.

Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a point in $\mathbb{C}^n$ with distinct coordinates. Introduce linear operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ on $V^{\otimes n}$ by the formula

$$\mathcal{H}_i(\boldsymbol{z}) = \sum_{j,\ j \neq i} \frac{\sum_{a,b=1}^{r+1} E_{a,b}^{(i)} \otimes E_{b,a}^{(j)}}{z_i - z_j}.$$

The operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ are called the **(quadratic) Gaudin Hamiltonians** of the Gaudin model associated with $V^{\otimes n}$.

The operator $\sum_{a,b=1}^{r+1} E_{a,b}^{(i)} \otimes E_{b,a}^{(j)}$ acts on $V^{\otimes n}$ simply by permuting the i-th and j-th factors.

Gaudin model

Proposition

The Gaudin Hamiltonians commute, $[\mathcal{H}_i(z), \mathcal{H}_j(z)] = 0$ *for all* i, j*. The Gaudin Hamiltonians commute with the action of* $\mathfrak{gl}_{r+1}$, $[\mathcal{H}_i(z), X] = 0$ *for all* i *and* $X \in \mathfrak{gl}_{r+1}$*. In particular, for any* $\mu \in \mathfrak{h}^*$*, the Gaudin Hamiltonians preserve the subspace* $(V^{\otimes n})^{\mathrm{sing}}_{\mu} \subset V^{\otimes n}$.

The main problem for the Gaudin model is to find common eigenvectors and eigenvalues of the Gaudin Hamiltonians. By the proposition, it suffices to do that in the subspace $(V^{\otimes n})^{\mathrm{sing}}_{\mu}$.

The main method is the **algebraic Bethe Ansatz method**.

What is Bethe Ansatz method?

The algebraic Bethe Ansatz method is a certain construction of eigenvectors for a family of commuting operators, which contains the Gaudin Hamiltonians.

The idea of this construction is to find a vector-valued function of a special form and determine its arguments in such a way that the value of this function is an eigenvector.

The function is called **weight function**. The equations which determine the special values of arguments are called the **Bethe Ansatz equations**.

The method was invented by [Bethe 1931] to find the exact spectrum of the one-dimensional antiferromagnetic Heisenberg model.

Example of case $\mathfrak{gl}_2$ [Gaudin 1976]

Consider the tensor product

$$(\mathbb{C}^2)^{\otimes n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

and introduce the generating function

$$\mathcal{K}(u) = \sum_{i=1}^{n} \frac{\mathcal{H}_i(\boldsymbol{z})}{u - z_i}, \quad \text{for } u \in \mathbb{C}.$$

Then $[\mathcal{K}(u), \mathcal{K}(w)] = 0$ for all $u, w \in \mathbb{C}$.

To diagonalize operators $\mathcal{K}(u)$ for all $u \in \mathbb{C}$ is equivalent to simultaneous diagonalization of the Gaudin Hamiltonians.

Note that

$$\mathcal{H}_i(z) = \sum_{j\neq i} \frac{E_{1,2}^{(i)} \otimes E_{2,1}^{(j)} + E_{2,1}^{(i)} \otimes E_{1,2}^{(j)} + E_{1,1}^{(i)} \otimes E_{1,1}^{(j)} + E_{2,2}^{(i)} \otimes E_{2,2}^{(j)}}{z_i - z_j},$$

the vector $|0\rangle := v_+ \otimes \cdots \otimes v_+ \in (\mathbb{C}^2)^{\otimes n}$ is an eigenvector of the Gaudin Hamiltonians; $\mathcal{H}_i(z)|0\rangle = \left(\sum_{j\neq i} \frac{1}{z_i - z_j}\right)|0\rangle$, therefore $|0\rangle$ is an eigenvector of $\mathcal{K}(u)$.

Introduce operators $E_{2,1}(t),\ t \in \mathbb{C}$, on the spaces $(\mathbb{C}^2)^{\otimes n}$ by the formula

$$E_{2,1}(t) = \sum_{i=1}^{n} \frac{E_{2,1}^{(i)}}{t - z_i}.$$

Now consider the vector

$$|t_1, \ldots, t_l\rangle = E_{2,1}(t_1) \ldots E_{2,1}(t_l)|0\rangle.$$

It is proved by explicit calculation in [Gaudin 1976] that

$$\mathcal{K}(u)|t_1,\dots,t_l\rangle = s_l(u)|t_1,\dots,t_l\rangle + \sum_{j=1}^{l} \frac{k_j}{u-t_j}|t_1,\dots,t_{j-1},u,t_{j+1},\dots,t_l\rangle,$$

where

$$k_j = \sum_{i=1}^{n} \frac{1}{t_j - z_i} - \sum_{s\neq j} \frac{2}{t_j - t_s},$$

$$s_l(u) = \sum_{1\leqslant i<j\leqslant n} \frac{1}{(u-z_i)(u-z_j)} - \sum_{i=1}^{n}\sum_{j=1}^{l} \frac{1}{(u-z_i)(u-t_j)} + \sum_{1\leqslant i<j\leqslant l} \frac{2}{(u-t_i)(u-t_j)}.$$

If we take $t_1, \ldots, t_l$ such that $k_j = 0$ for all $j = 1, \ldots, l$, then the vector $|t_1, \ldots, t_l\rangle$ becomes an eigenvector of $\mathcal{K}(u)$. The system of equations $k_j = 0$ for all $j = 1, \ldots, l$ is called the **Bethe Ansatz equation (BAE)**. The vector $|t_1, \ldots, t_l\rangle$ is called a **Bethe vector**.

By computing the residues, we get the corresponding eigenvalues of $\mathcal{H}_i(z)$, that is

$$\mathcal{H}_i(z)|t_1, \ldots, t_l\rangle = \Big(\sum_{j\neq i} \frac{1}{z_i - z_j} - \sum_{j=1}^{l} \frac{1}{z_i - t_j}\Big)|t_1, \ldots, t_l\rangle.$$

Moreover, the residue of $s_l(u)$ at $u = t_j$ is $k_j = 0$ for each j. Hence $s_l(u)$ is regular at t_j for all $j = 1, \ldots, l$.

Master function for $\mathfrak{gl}_2$

We introduce a rational function $\Phi(\boldsymbol{t};\boldsymbol{z})$, where $\boldsymbol{t}=(t_1,\dots,t_l)$ and $\boldsymbol{z}=(z_1,\dots,z_n)$, by the formula

$$\Phi(\boldsymbol{t};\boldsymbol{z})=\prod_{1\leqslant i<j\leqslant n}(z_i-z_j)\prod_{i=1}^{l}\prod_{j=1}^{n}(t_i-z_j)^{-1}\prod_{1\leqslant i<j\leqslant l}(t_i-t_j)^2.$$

Then we have

$$k_i=\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_i}\right)(\boldsymbol{t};\boldsymbol{z}),$$

$$\mathcal{H}_i(\boldsymbol{z})|t_1,\dots,t_l\rangle=\left(\Phi^{-1}\frac{\partial\Phi}{\partial z_i}\right)(\boldsymbol{t};\boldsymbol{z})|t_1,\dots,t_l\rangle,$$

if $\boldsymbol{t}=(t_1,\dots,t_l)$ satisfies BAE. Moreover, with the standard inner product on $(\mathbb{C}^2)^{\otimes n}$, we have

$$(|t_1,\dots,t_l\rangle,|t_1,\dots,t_l\rangle)=\det_{1\leqslant i,j\leqslant l}\left(\frac{\partial^2}{\partial t_i\partial t_j}\ln\Phi(\boldsymbol{t};\boldsymbol{z})\right).$$

MASTER FUNCTION FOR $\mathfrak{gl}_{r+1}$

Fix a sequence of non-negative integers $\boldsymbol{l} = (l_1, \dots, l_r)$. Denote $l = l_1 + \cdots + l_r$, $\alpha(\boldsymbol{l}) = l_1\alpha_1 + \cdots + l_r\alpha_r$, $\Lambda_\infty = n\omega_1 - \alpha(\boldsymbol{l})$. Introduce the **master function** $\Phi(\boldsymbol{t}; \boldsymbol{z})$ which is a function of l variables

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}; \dots; t_1^{(r)}, \dots, t_{l_r}^{(r)})$$

by the formula

$$\Phi(\boldsymbol{t}; \boldsymbol{z}) = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \prod_{j=1}^{l_1} \prod_{s=1}^{n} (t_j^{(1)} - z_s)^{-1}$$

$$\times \prod_{i=1}^{r} \prod_{1 \leqslant j < s \leqslant l_i} (t_j^{(i)} - t_s^{(i)})^2 \prod_{i=1}^{r-1} \prod_{j=1}^{l_i} \prod_{k=1}^{l_{i+1}} (t_j^{(i)} - t_k^{(i+1)})^{-1}$$

Bethe Ansatz equation

A point $\boldsymbol{t} \in \mathbb{C}^l$ is called a **critical point** of $\Phi(\cdot\,;\boldsymbol{z})$, if

$$\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t};\boldsymbol{z}) = 0, \quad \text{for } i = 1,\dots,r \text{ and } j = 1,\dots,l_i.$$

In other words, the following system of algebraic equations is satisfied,

$$0 = \sum_{s=1}^{n}\frac{\delta_{1,i}}{t_j^{(i)} - z_s} - \sum_{s=1,s\neq j}^{l_i}\frac{2}{t_j^{(i)} - t_s^{(i)}} + \sum_{s=1}^{l_{i+1}}\frac{1}{t_j^{(i)} - t_s^{(i+1)}} + \sum_{s=1}^{l_{i-1}}\frac{1}{t_j^{(i)} - t_s^{(i-1)}}.$$

This system of the equations is called the **Bethe Ansatz equation** associated to $\mathfrak{gl}_{r+1}$ Gaudin model.

The product of symmetric group $\boldsymbol{\Sigma_l} = \Sigma_{l_1} \times \cdots \times \Sigma_{l_r}$ acts on the variables $\boldsymbol{t}$ by permuting the coordinates with the same upper index. The master function is $\boldsymbol{\Sigma_l}$-invariant. The set of critical points of $\Phi(\cdot\,;\boldsymbol{z})$ is $\boldsymbol{\Sigma_l}$-invariant. We will not distinguish critical points in the same $\boldsymbol{\Sigma_l}$-orbit.

For a critical point $\boldsymbol{t}$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \ldots, y_r)$ of polynomials of x by

$$y_i(x) = \prod_{j=1}^{l_i} (x - t_j^{(i)}) \quad \text{for } i = 1, \ldots, r.$$

Solving BAE is equivalent to finding coefficients of y_i. We say that $\boldsymbol{y}^{\boldsymbol{t}}$ **represent the critical point $\boldsymbol{t}$**.

General master function

Fix a simple Lie algebra $\mathfrak{g}$, a sequence of weights $\boldsymbol{\Lambda} = (\Lambda_i)_{i=1}^n$, $\Lambda_i \in \mathfrak{h}^*$, and a sequence of non-negative integers $\boldsymbol{l} = (l_1, \dots, l_r)$. Denote $L_{\boldsymbol{\Lambda}} = L_{\Lambda_1} \otimes L_{\Lambda_2} \otimes \cdots \otimes L_{\Lambda_n}$, $l = l_1 + \cdots + l_r$, $\alpha(\boldsymbol{l}) = l_1\alpha_1 + \cdots + l_r\alpha_r$, and $\Lambda_\infty = \Lambda_1 + \cdots + \Lambda_n - \alpha(\boldsymbol{l})$. Introduce the **master function** $\Phi_{\mathfrak{g},\boldsymbol{\Lambda},\boldsymbol{l}}(\boldsymbol{t};\boldsymbol{z})$ which is a function of l variables

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}; \dots; t_1^{(r)}, \dots, t_{l_r}^{(r)})$$

by the formula

$$\Phi_{\mathfrak{g},\boldsymbol{\Lambda},\boldsymbol{l}}(\boldsymbol{t};\boldsymbol{z}) = \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{(\Lambda_i,\Lambda_j)} \prod_{i=1}^{r}\prod_{j=1}^{l_i}\prod_{s=1}^{n}(t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)}$$
$$\times \prod_{i=1}^{r}\prod_{1\leqslant j<s\leqslant l_i}(t_j^{(i)} - t_s^{(i)})^{(\alpha_i,\alpha_i)} \prod_{1\leqslant i<j\leqslant r}\prod_{s=1}^{l_i}\prod_{k=1}^{l_j}(t_s^{(i)} - t_k^{(j)})^{(\alpha_i,\alpha_j)}.$$

Weight Function

The formula for the Bethe vector is a rational map

$$\omega : \mathbb{C}^l \times \mathbb{C}^n \to (V^{\otimes n})_{\Lambda_\infty}, \quad (\boldsymbol{t}, \boldsymbol{z}) \mapsto \omega(\boldsymbol{t}; \boldsymbol{z})$$

called the **canonical weight function**, which was introduced by [Matsuo 1990] for $\mathfrak{gl}_{r+1}$ and by [Schechtman-Varchenko 1991] for all simple Lie algebras.

Let $\boldsymbol{t} \in \mathbb{C}^l$ be a critical point of the master function $\Phi(\cdot\,; \boldsymbol{z})$. Then the value of the weight function $\omega(\boldsymbol{t}\,; \boldsymbol{z}) \in (V^{\otimes n})_{\Lambda_\infty}$ is called the **Bethe vector**.

Note that $\dim L_\lambda < \infty$ if and only if $\lambda \in \mathfrak{h}^*$ is **dominant integral**.

Lemma (Mukhin-Varchenko 2004)

If weight Λ_∞ is dominant integral, then the set of critical points is finite.

Assume that $\boldsymbol{t} \in \mathbb{C}^l$ is an isolated critical point of the master function $\Phi(\cdot\,;\boldsymbol{z})$. Then with the standard inner product $(\cdot\,,\cdot\,)$,

$$(\omega(\boldsymbol{t};\boldsymbol{z}),\omega(\boldsymbol{t};\boldsymbol{z})) = \det_{1\leqslant i,j\leqslant l}\left(\frac{\partial^2}{\partial t_i \partial t_j}\ln\Phi(\boldsymbol{t};\boldsymbol{z})\right).$$

This equality is proved by [Varchenko 2006] for the general setting.

Theorem (Varchenko 2011)

The Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is non-zero.

Theorem (Reshetikhin-Varchenko 1995)

The Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is singular, $\omega(\boldsymbol{t};\boldsymbol{z}) \in (V^{\otimes n})^{\mathrm{sing}}_{\Lambda_\infty}$. Moreover, $\omega(\boldsymbol{t};\boldsymbol{z})$ is a common eigenvector of the Gaudin Hamiltonians,

$$\mathcal{H}_i(\boldsymbol{z})\omega(\boldsymbol{t};\boldsymbol{z}) = \left(\Phi^{-1}\frac{\partial\Phi}{\partial z_i}\right)(\boldsymbol{t};\boldsymbol{z})\omega(\boldsymbol{t};\boldsymbol{z}), \qquad i = 1,\ldots,n.$$

Completeness

Bethe Ansatz Conjecture

The number of solutions of BAE equals to $\dim(V^{\otimes n})^{\text{sing}}_{\Lambda_\infty}$ *and the Bethe vectors obtained from those solutions form a basis of* $(V^{\otimes n})^{\text{sing}}_{\Lambda_\infty}$.

This conjecture is true for generic z. For general setting, the conjecture is **false**. In [Mukhin-Varchenko 2007], there is a counterexample for which the conjecture is false for all z. When z is generic, the conjecture is true for the following cases with certain tensor products

- Lie algebra $\mathfrak{gl}_{r+1}$ [Mukhin-Varchenko 2005]
- Lie superalgebra $\mathfrak{gl}(p|q)$ [Mukhin-Vicedo-Young 2015]
- Lie algebras of types B,C,D [L-Mukhin-Varchenko 2015]

Completeness for types B,C,D

The starting point is the tensor products of the first fundamental representations L_{ω_1}. The L_{ω_1} plays the similar role as V in $\mathfrak{gl}_{r+1}$. To this end, we begin with the 2-point case $L_\lambda \otimes L_{\omega_1}$.

For 2-point case, we can always rescale (z_1, z_2) to $(0, 1)$. Note that the decomposition of $L_\lambda \otimes L_{\omega_1}$ is **multiplicity-free** for all dominant integral weights λ. We expect to solve the BAE explicitly.

This is equivalent to finding the coefficients of a tuple of polynomials $\boldsymbol{y}^t$, which represents this solution. In all previously known results for the multiplicity-free cases, those coefficients can be completely factorized into products of linear functions of the parameters with integer coefficients. The difficulty for types B,C,D is that the coefficients can not be factorized in such a fashion.

Our idea comes from the reproduction procedure studied in [Mukhin-Varchenko 2008]. This reproduction procedure allows us to reduce the problem to the trivial case $\boldsymbol{l} = (0, \dots, 0)$ with different initial data. By solving the BAE, we obtain that the constant terms of those polynomials whose degrees are at most 2 are factorizable while the linear coefficients can be expressed as sums of two factorizable terms.

Theorem (L-Mukhin-Varchenko 2015)

The set of Bethe vectors $\omega(\boldsymbol{t}; z_1, z_2)$, where $\boldsymbol{t}$ runs over the solutions to the Bethe Ansatz equations with $\boldsymbol{l}$, forms a basis of $(L_\lambda \otimes L_{\omega_1})^{\mathrm{sing}}$. Moreover, for types B and C, the Gaudin Hamiltonians have simple joint spectrum.

The spectrum is not simple for type D since the Dynkin diagram for type D has a nontrivial symmetry.

Let $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$. For the case $\boldsymbol{l} = (2, \ldots, 2)$, the constant term of the quadratic polynomial y_k, $k = 1, \ldots, r$, is

$$\left(\prod_{j=1}^{2r-l} \frac{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j} \right)^2$$

$$\times \prod_{j=2r-l+1}^{r-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}$$

$$\times \prod_{j=2r-l+1}^{k-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}$$

$$\times \prod_{i=1}^{r-k} \frac{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i - 1}{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i}$$

$$\times \frac{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 3}{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 1}.$$

The linear coefficient of the quadratic polynomial y_k is

$$\frac{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-3}{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-2}$$
$$\times\prod_{j=1}^{2r-l}\frac{\lambda_j+\cdots+\lambda_{2r-l}+2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}{\lambda_j+\cdots+\lambda_{2r-l}+2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j}$$
$$\times\left(\prod_{j=2r-l+1}^{k-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-2}{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}\right.$$
$$+\prod_{j=2r-l+1}^{r-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-2}{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}$$
$$\left.\times\prod_{i=1}^{r-k}\frac{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i-1}{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i}\right).$$

We can reduce the completeness for $L_\lambda \otimes \left(L_{\omega_1}^{\otimes n}\right)$ case to the case of $n = 2$ by sending all z_i to the same number with different rates. We can generalize the theorem to the general n, that is for the tensor product $L_\lambda \otimes \left(L_{\omega_1}^{\otimes n}\right)$ for all dominant integral weight λ.

Theorem (L-Mukhin-Varchenko 2015)

Let $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{so}_{2r}$ or $\mathfrak{sp}_{2r}$. For generic $\boldsymbol{z}$ there exists a set of solutions $\{\boldsymbol{t}_i,\ i \in I\}$ of the Bethe Ansatz equation such that the corresponding Bethe vectors $\{\omega(\boldsymbol{t}_i; \boldsymbol{z}),\ i \in I\}$ form a basis of $\left(L_\lambda \otimes L_{\omega_1}^{\otimes n}\right)^{\mathrm{sing}}$. If $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{sp}_{2r}$, the Gaudin Hamiltonians have simple joint spectrum.

LIE ALGEBRA
- Lie algebra $\mathfrak{gl}_{r+1}$

BETHE ANSATZ IN GAUDIN MODEL
- Gaudin model
- Bethe Ansatz
- Weight function

COMPLETENESS
- Obatained results for completeness of types B,C,D

CONNECTIONS AND PROPOSED RESEARCH
- Selberg integral
- Orthogonal polynomial
- Fuchsian differential operators
- Schubert calculus

Selberg Integral

Consider the case $\mathfrak{gl}_2$, $n = 2$, $\boldsymbol{\Lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{z} = (0, 1)$. Then for given l, we have

$$\Phi(\boldsymbol{t}; \boldsymbol{z}) = \prod_{j=1}^{l} t_j^{-\lambda_1}(1 - t_j)^{-\lambda_2} \prod_{1 \leqslant i < j \leqslant l} (t_i - t_j)^2.$$

The Selberg integral is

$$\int_\Delta \Phi^{\frac{1}{\kappa}} d\boldsymbol{t} = \prod_{j=0}^{l-1} \frac{\Gamma((-\lambda_1 + j)/\kappa + 1)\Gamma((-\lambda_2 + j)/\kappa + 1)\Gamma((j+1)/\kappa + 1)}{(j+1)\Gamma((-\lambda_1 - \lambda_2 + (2l - j - 2))/\kappa + 2)\Gamma(1/\kappa + 1)},$$

where $\Delta = \{t \in \mathbb{R}^l \mid 0 < t_1 < \cdots < t_l < 1\}$.

The following is a conjecture made by [Mukhin-Varchenko 2000] for the case of tensor product of two highest weight modules.

If $\dim(L_\lambda \otimes L_\mu)^{\text{sing}}_{\Lambda_\infty} = 1$, then the integral $\int \Phi^{1/\kappa} d\boldsymbol{t}$ for some region can be computed explicitly in terms of Gamma functions.

There is a 43 pages paper written by [Forrester-Warnaar 2007] about the importance of Selberg integral. There are several results in this direction:

- for the $\mathfrak{sl}_3$ case [Tarasov-Varchenko 2003];
- for the $\mathfrak{sl}_{r+1}$ case [Warnaar 2009];
- for the B,C,D case with $L_{\omega_1} \otimes L_{\omega_1}$ [Mimachi-Takamuki 2004].

We hope to compute this integral for type B (and C,D) with $L_\lambda \otimes L_{\omega_1}$.

EXAMPLE [THEOREM 6.7.1, SZEGÖ]

The Jacobi polynomial $P_l^{(a,b)}(t)$ satisfies the following differential equation

$$(1-t^2)y'' + (b-a-t(a+b+2))y' + l(a+b+l+1)y = 0.$$

The roots $t_1, \ldots, t_l$ of $P_l^{(a,b)}(t)$ are simple and satisfy the following system of equations

$$\frac{a+1}{t_j-1} + \frac{b+1}{t_j+1} + \sum_{k\neq j} \frac{2}{t_j-t_k} = 0,$$

for any $j = 1, \ldots, l$.

This is the Bethe Ansatz equation associated to $\mathfrak{gl}_2$, $\boldsymbol{z} = (1, -1)$, $L_{-a-1} \otimes L_{-b-1}$, $\boldsymbol{l} = (l)$.

Orthogonal polynomial

The previous example explains how the solutions of BAE for the $\mathfrak{gl}_2$ case are related to the **Jacobi polynomials**.

The Jacobi polynomials are orthogonal polynomials,

$$\int_{-1}^{1} P_m^{(a,b)}(x) P_n^{(a,b)}(x)(1-x)^a(1+x)^b dx$$

$$= \frac{2^{a+b+1}}{2m+a+b+1} \frac{\Gamma(m+a+1)\Gamma(m+b+1)}{\Gamma(m+a+b+1)m!} \delta_{n,m}, \quad a,b > -1.$$

For the case of $\mathfrak{sl}_{r+1}$, $\boldsymbol{\Lambda} = (\Lambda, k\omega_1)$, where $k \in \mathbb{Z}_{\geqslant 0}$, the solutions of the Bethe Ansatz equations are related to zeros of **Jacobi-Piñeiro polynomials** which are multiple orthogonal polynomials, [Mukhin-Varchenko 2007].

Fuchsian differential operators

Let $\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(n)}, \boldsymbol{\lambda}$ be partitions of length at most $r+1$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{r+1})$ and $\boldsymbol{\lambda}^{(s)} = (\lambda_1^{(s)}, \dots, \lambda_{r+1}^{(s)})$ for all $s = 1, \dots, n$. Set $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(n)})$. Let $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},z}$ be the set of all monic Fuchsian differential operators of order $r+1$,

$$\mathcal{D} = \partial_u^{r+1} + \sum_{i=1}^{r+1} h_i^{\mathcal{D}}(u)\partial_u^{r+1-i},$$

with the following properties:

- The singular points of $\mathcal{D}$ are $z_1, \dots, z_n$ and ∞.
- The exponents of $\mathcal{D}$ at z_s, $s = 1, \dots, n$, are $\lambda_{r+1}^{(s)}, \lambda_r^{(s)} + 1, \dots, \lambda_1^{(s)} + r$.
- The exponents of $\mathcal{D}$ at ∞, $s = 1, \dots, n$, are $-r - \lambda_1, 1 - r - \lambda_2, \dots, -\lambda_{r+1}$.
- The operator $\mathcal{D}$ has no monodromy.

higher Gaudin Hamiltonians

Consider $\mathfrak{gl}_{r+1}$ Gaudin model with $\mathfrak{gl}_{r+1}$ modules $L_{\boldsymbol{\lambda}^{(s)}}$, $s = 1, \dots, n$.

There exist commuting operators $B_{i,j}$ in $\mathrm{End}(\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})$, for all $i = 1, \dots, r+1$, $j \in \mathbb{Z}_{\geqslant i}$. The operators $B_{i,j}$ are called **higher Gaudin Hamiltonians**.

Set $B_i(u) = \sum_{j=i}^{\infty} B_{ij} u^{-j}$, then we have

$$B_2(u) = \sum_{i=1}^{n} \frac{\mathcal{H}_i(z_i)}{u - z_i} + \sum_{i=1}^{n} \frac{k_{\boldsymbol{\lambda}^{(i)}}}{(u - z_i)^2},$$

$k_{\boldsymbol{\lambda}^{(i)}}$ is a constant related to the action of the Casimir element on $L_{\boldsymbol{\lambda}^{(i)}}$.

Theorem (Mukhin-Tarasov-Varchenko 2009)

There exists a bijection between the set of common eigenvectors of the action of $B_{i,j}$ *on* $(\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})^{\text{sing}}_{\boldsymbol{\lambda}}$ *and the set* $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{z}}$*. More specifically, for each common eigenvector* v*, denote* $B_i(u)v = h_i(u)v$ *then*

$$\partial_u^{r+1} + \sum_{i=1}^{r+1} h_i(u)\partial_u^{r+1-i} \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}.$$

Moreover, $B_{i,j}$ *have simple joint spectrum (for all* $\boldsymbol{z}$*) and they generate a maximal commutative subalgebra of dimension* $\dim(\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})^{\text{sing}}_{\boldsymbol{\lambda}}$ *in* $\text{End}(\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})$.

Schubert calculus

Set $\bar{\boldsymbol{\lambda}} = (d - r - 1 - \lambda_{r+1}, \ldots, d - r - 1 - \lambda_1)$ for enough large $d \in \mathbb{Z}_{\geqslant 0}$. We define the intersection of Schubert cells $\Omega_{\boldsymbol{\Lambda}, \bar{\boldsymbol{\lambda}}, z}$ on the Grassmannian of $\mathbb{C}_d[u]$. The space of polynomials X is a point of $\Omega_{\boldsymbol{\Lambda}, \bar{\boldsymbol{\lambda}}, z}$ if and only if X is the kernel of a differential operator in $\Delta_{\boldsymbol{\Lambda}, \boldsymbol{\lambda}, z}$. We call $X \in \Omega_{\boldsymbol{\Lambda}, \bar{\boldsymbol{\lambda}}, z}$ **real** if X has a basis consisting of polynomials with real coefficients. Equivalently, a point $\mathcal{D} \in \Delta_{\boldsymbol{\Lambda}, \boldsymbol{\lambda}, z}$ is **real** if all $h_i^{\mathcal{D}}(x)$ are rational functions with real coefficients.

Theorem (Mukhin-Tarasov-Varchenko 2009)

Let $z_1, \ldots, z_n$ be distinct real numbers. Then $\Omega_{\boldsymbol{\Lambda}, \bar{\boldsymbol{\lambda}}, z}$ consist of $\dim(\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ distinct real points.

This is called the **strong form of the B. and M. Shapiro conjecture**. In particular, the theorem implies the transversality of the intersection. This statement was a long-standing conjecture, see [Eisenbud-Harris 1986] and [Sottile 2000].

It is proved that for types B,C the solutions of BAE also correspond to Fuchsian differential operators, see [Mukhin-Varchenko 2004]. In the cases of types B,C, we can associate each critical point to a differential operator whose kernel is a vector space of polynomials with certain symmetry. Such a space is called **self-dual space**. We hope to understand the geometric property of this subset consisting of all the dual spaces.

It is also proved that the coefficients of the constructed differential operator correspond to the eigenvalues of higher Gaudin Hamiltonians with respect to the corresponding Bethe vector for types B,C, see [Mukhin-Molev 2015]. This result gives hope of applications of Gaudin models of types B,C in geometry. The long term goal is to establish the similar statements as the previous theorems for types B, C. In particular, we would like to prove the (higher) Gaudin Hamiltonians have simple joint spectrum for all z.

References

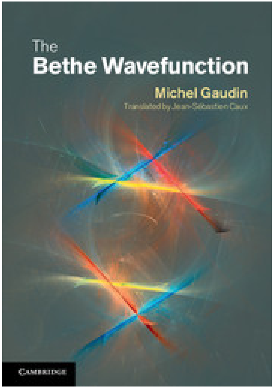
- H. Bethe, *Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette*, Zeitschrift fr Physik (1931), 71:205–226 .
- B. Feigin, E. Frenkel, N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys., **166** (1994), no. 1, 2762.
- P. Forrester, S. Warnaar, *The importance of the Selberg integral*, Bull. Amer. Math. Soc. 45 (2008), 489–534.
- M. Gaudin, *Diagonalisation dune classe dHamiltoniens de spin*, J. Physique, **37** (1976), 1087-1098.
- M. Gaudin, La fonction donde de Bethe, Collection du Commissariat l Ergie Atomique, Serie Scientifique, Masson, Paris, 1983.
- K. Lu, E. Mukhin, A. Varchenko, *On the Gaudin model associated to Lie algebras of classical types*, preprint, math.QA/1512.08524.
- A. Molev, *FeiginFrenkel center in types B, C and D*, Invent. Math., **191** (2003), no. 1, 1–34.
- A. Molev, E. Mukhin, *Eigenvalues of Bethe vectors in the Gaudin model*, preprint, math.RT/1506.01884.

- K. Mimachi and T. Takamuki, *A generalization of the beta integral arising from the Knizhnik Zamolodchikov equation for the vector representations of types B_n, C_n and D_n*, Kyushu J. Math. **59** (2005), 117126.
- E. Mukhin, V. Tarasov, A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Ann. Math. **170** (2009), no. 2, 863–881, math.AG/0512299.
- E. Mukhin, V. Tarasov, A. Varchenko, *Schubert calculus and representations of general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940, math.QA/0711.4079.
- E. Mukhin, A. Varchenko, *Remarks on critical points of phase functions and norms of Bethe vectors*, In: Arrangements–Tokyo 1998, Adv. Studies in Pure Math. **27** (2000), 239–246, math.RT/9810087.
- E. Mukhin, A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), no. 1, 111–163, math.QA/0209017.
- E. Mukhin, A. Varchenko, *Norm of a Bethe vector and the Hessian of the master function*, Compos. Math. **141** (2005), no. 4, 1012-1028, math.QA/0402349.

- E. Mukhin, A. Varchenko, *Multiple orthogonal polynomials and a counterexample to Gaudin Bethe Ansatz Conjecture*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5383–5418, math.QA/0501144.
- E. Mukhin, A. Varchenko, *Quasi-polynomials and the Bethe ansatz*, Geom. Topol. Monogr. **13** (2008), 385–420, math.QA/0604048.
- E. Mukhin, B. Vicedo, C. Young, *Gaudin Model for* $\mathfrak{gl}(m|n)$, J. Math. Phys. **56** (2015), no. 5, 051704, math.QA/1404.3526.
- N. Reshetikhin, A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, Geometry, Topology and Physics for R. Bott, Intern. Press, (1995), 293C-322, hep-th/9402126.
- G. Szegö, Orthogonal Polynomials, 4-th edition, American Mathematical Society, 1939.
- V. Schechtman, A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991) 139-194.
- D. Talalaev, *Quantization of the Gaudin system*, Funct. Anal. Appl., **40** (2006), no. 1, 86–91.

- A. Varchenko, *Bethe Ansatz for arrangements of hyperplanes and the Gaudin model*, Mosc. Math. J. **6** (2006), no. 1, 195–210, math.QA/0408001.
- A. Varchenko, *Quantum integrable model of an arrangement of hyperplanes*, SIGMA Symmetry Integrability Geom. Methods Appl. **7** (2011), Paper 032, 1–55, math.QA/1001.4553.
- V. Tarasov, A. Varchenko, *Selberg-Type Integrals Associated with* $\mathfrak{sl}_3$, Lett. Math. Phys., **65** (2003), no. 3, 173–185.
- S. Warnaar, *A Selberg integral for the Lie algebra* A_n, Acta Math., **203** 2009, 269–304.
- A. Eremenko and A. Gabrielov, *Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry*, Ann. Math. (2) **155** (2002), no. 1, 105-129.
- F. Sottile, *Rational curves on Grassmannians: systems theory, reality, and transversality*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math., 276, Amer. Math. Soc., Providence, RI (2001), 9-42.


Bethe, Hans
K 3


The
Bethe Wavefunction
Michel Gaudin
Translated by Jean-Sébastien Caux
CAMBRIDGE

Thank you!

Q & A

Why vector representation?

Solving Bethe Ansatz equation for arbitrary case is impossible. But it is possible to solving it for tensor product of two representation with multiplicity-free decomposition. The BAE for vector representation is relatively easy to solve.

The Schur-Weyl duality tells us that every finite-dimensional irreducible representation is included in $V^{\otimes n}$ when n is large enough. Though we can't solve BAE for the tensor product of arbitrary finite-dimensional irreducible representations, we can collide some z_i making them equal and use the result from vector representation to deduce results for more general cases with generic $\boldsymbol{z}$.

This also suggests us to solve BAE for the tensor products of the first fundamental representations for types B,C,D as a first step.

WEIGHT FUNCTION

In 1991, **Schechtman** and **Varchenko** defined a rational map

$$\omega : \mathbb{C}^l \times \mathbb{C}^n \to (L_{\boldsymbol{\Lambda}})_{\Lambda_\infty}, \quad (\boldsymbol{t}, \boldsymbol{z}) \mapsto \omega(\boldsymbol{t}; \boldsymbol{z})$$

called **the canonical weight function**. Let $P(\boldsymbol{l}, n)$ be the set of sequences $I = (i_1^1, \ldots, i_{j_1}^1; \ldots; i_1^n, \ldots, i_{j_n}^n)$ of integers in $\{1, \ldots, r\}$ such that for all $i = 1, \ldots, r$, the integer i appears in I precisely l_i times. To every $I \in P(\boldsymbol{l}, n)$ we associate a vector

$$E_I v = E_{i_1^1+1, i_1^1} \ldots E_{i_{j_1}^1+1, i_{j_1}^1} v_+ \otimes \cdots \otimes E_{i_1^n+1, i_1^n} \ldots E_{i_{j_n}^n+1, i_{j_n}^n} v_+$$

in $(L_{\boldsymbol{\Lambda}})_{\Lambda_\infty}$, and certain rational functions ω_I. We set

$$\omega(\boldsymbol{t}; \boldsymbol{z}) = \sum_{I \in P(\boldsymbol{l}, n)} \sum_{\text{sym}} \omega_I E_I v.$$

Example of Weight Function

Consider the vector representation of $\mathfrak{gl}_{r+1}$. If $\boldsymbol{l} = (1, 1, 0, \dots, 0)$, then

$$\begin{aligned}
\omega(\boldsymbol{t}; \boldsymbol{z}) = {} & \frac{E_{2,1}E_{3,2}v_+ \otimes v_+}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_1)} + \frac{E_{3,2}E_{2,1}v_+ \otimes v_+}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_1)} \\
& + \frac{E_{2,1}v_+ \otimes E_{3,2}v_+}{(t_1^{(1)} - z_1)(t_1^{(2)} - z_2)} + \frac{E_{3,2}v_+ \otimes E_{2,1}v_+}{(t_1^{(2)} - z_1)(t_1^{(1)} - z_2)} \\
& + \frac{v_+ \otimes E_{2,1}E_{3,2}v_+}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_2)} + \frac{v_+ \otimes E_{3,2}E_{2,1}v_+}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_2)} \\
= {} & \frac{E_{3,2}E_{2,1}v_+ \otimes v_+}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_1)} + \frac{v_+ \otimes E_{3,2}E_{2,1}v_+}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_2)}.
\end{aligned}$$

Examples of Weight Function

If $\boldsymbol{l} = (2, 0, \ldots, 0)$, then

$$
\begin{aligned}
&\omega(\boldsymbol{t}; \boldsymbol{z}) \\
&= \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_1)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_1)} \Big) E_{2,1}^2 v_+ \otimes v_+ \\
&+ \Big(\frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - z_1)(t_1^{(1)} - z_2)} \Big) E_{2,1} v_+ \otimes E_{2,1} v_+ \\
&+ \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_2)} \Big) v_+ \otimes E_{2,1}^2 v_+ \\
&= \Big(\frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - z_1)(t_1^{(1)} - z_2)} \Big) E_{2,1} v_+ \otimes E_{2,1} v_+ .
\end{aligned}
$$

CONSTRUCTION OF A DIFFERENTIAL OPERATOR FROM A CRITICAL POINT?

Consider the Gaudin model associated to the tensor produc of vector representations of $\mathfrak{gl}_{r+1}$. Let $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \dots, y_r)$ represent a critical point $\boldsymbol{t}$. Define

$$D(\boldsymbol{y}) = \left(\partial - \ln' \frac{T_1}{y_r}\right)\left(\partial - \ln' \frac{T_1 y_r}{y_{r-1}}\right) \cdots \left(\partial - \ln' \frac{y_2 T_1}{y_1}\right)(\partial - \ln' y_1),$$

where $T_1(x) = (x - z_1) \cdots (x - z_n)$.

Higher Gaudin Hamiltonians

Given an $N \times N$ matrix A with possibly non-commuting entries a_{ij}, we define its **row determinant** to be

$$\operatorname{rdet} A = \sum_{\sigma \in S_N} (-1)^{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{N\sigma(N)}.$$

Define the differential operator $\mathcal{D}^{\mathcal{B}}$ by

$$\operatorname{rdet} \begin{pmatrix} \partial_u - \sum_{s=1}^{n} \frac{E_{1,1}^{(s)}}{u - z_s} & - \sum_{s=1}^{n} \frac{E_{2,1}^{(s)}}{u - z_s} & \ldots & - \sum_{s=1}^{n} \frac{E_{r+1,1}^{(s)}}{u - z_s} \\ - \sum_{s=1}^{n} \frac{E_{1,2}^{(s)}}{u - z_s} & \partial_u - \sum_{s=1}^{n} \frac{E_{2,2}^{(s)}}{u - z_s} & \ldots & - \sum_{s=1}^{n} \frac{E_{r+1,2}^{(s)}}{u - z_s} \\ \ldots & \ldots & \ldots & \ldots \\ - \sum_{s=1}^{n} \frac{E_{1,r+1}^{(s)}}{u - z_s} & - \sum_{s=1}^{n} \frac{E_{2,r+1}^{(s)}}{u - z_s} & \ldots & \partial_u - \sum_{s=1}^{n} \frac{E_{r+1,r+1}^{(s)}}{u - z_s} \end{pmatrix}.$$

It is a differential operator in variable u, whose coefficients are formal power series in u^{-1} with coefficients in $\mathrm{End}\Big((\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})^{\mathrm{sing}}_{\boldsymbol{\lambda}}\Big)$,

$$\mathcal{D}^{\mathcal{B}} = \partial_u^{r+1} + \sum_{i=1}^{r+1} B_i(u)\partial_u^{r+1-i},$$

where $B_i(u) = \sum_{j=i}^{\infty} B_{ij}u^{-j}$ and $B_{ij} \in \mathrm{End}\Big((\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})^{\mathrm{sing}}_{\boldsymbol{\lambda}}\Big)$, for all $i = 1, \dots, r+1$, $j \in \mathbb{Z}_{\geqslant i}$. The operators $B_{i,j}$ are called the **higher Gaudin Hamiltonians**.

The higher Gaudin Hamiltonians for arbitrary simple Lie algebras were introduced by [Feigin-Frenkel-Reshetikhin 1994], see also [Molev 2013].

Theorem (Talalaev 2004)

The operators $B_{i,j}$ generate a commutative subalgebra in $\mathrm{End}\Big((\bigotimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}})^{\mathrm{sing}}_{\boldsymbol{\lambda}}\Big)$ and they commute with the action of $\mathfrak{gl}_{r+1}$.

The B. and M. Shapiro conjecture

If $V \subset \mathbb{C}[x]$ be a vector subspace of dimension $r+1$. The space V is called **real** if it has a basis consisting of real polynomials.

The B. and M. Shapiro conjecture: Let $f_{,} \dots, f_{r+1}$ be a basis of V. If all roots of the polynomial $\mathrm{Wr}(f_1, \dots, f_{r+1})$ are real, then the space is real.

Sketch of proof: First we reduce it to the case that all roots have order 1.Define a differential operator with kernel V. It corresponds to a Bethe vector in the tensor product of vector representations. Another important observation is that the higher Gaudin hamiltonians are symmetric with respect to a positive definite form(Shapovalov form). Note that the higher Gaudin hamiltonian is symmetric, it must have real eigenvalue. It follows that the coefficients of the operator are real.Q.E.D.

Self-dual space

Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(n)}, \boldsymbol{\lambda}$ be partitions with at most $r+1$ parts and $z_1, \ldots, z_n$ distinct complex numbers. For $U \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},z}$ and $g_1, \ldots, g_i \in U$, define a polynomial

$$\mathrm{Wr}^{\dagger}(g_1, \ldots, g_i) = \mathrm{Wr}(g_1, \ldots, g_i) \prod_{j=1}^{i} T_j^{j-i-1},$$

the **divided Wronskian with respect to X**, where T_i, $i = 1, \ldots, r+1$, are defined by

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\lambda^{(s)}_{r+2-i} - \lambda^{(s)}_{r+3-i}},$$

where $\lambda^{(s)}_{r+2} = 0$, for all $s = 1, \ldots, n$. Denote $U^{\dagger}$ the space spanned by $\mathrm{Wr}^{\dagger}(g_1, \ldots, g_r)$ for all $g_i \in U$. $U^{\dagger}$ is called the **dual space** of U. A space of polynomials U is called **self-dual** if $U = U^{\dagger}$.