Strong maximal inequality in L_p

Our main goal for this project is to show the maximal inequality on L^p with the constant $C_p = p/(p-1)$. That is, given a measure space $(\Omega, \mathcal{F}, \mu)$, any ergodic transformation τ on $(\Omega, \mathcal{F}, \mu)$ and $f \in L_p(\Omega)$, if we define

$$f^* = \sup_{n \geqslant 1} \left| \frac{f \circ \tau + \cdots + f \circ \tau^n}{n} \right|, \tag{1}$$

then

$$\|f^*\|_p \leqslant \frac{p}{p-1} \|f\|_p \quad \text{for all } p,\ 1 < p < \infty. \tag{2}$$

In order to transform the question to the case [1], we need the following lemma

Lemma 1. [3] *If $(\Omega, \mathcal{F}, \mu)$ is σ-finite, a measurable transformation T is measure-preserving if and only if all integrable nonnegative functions $f : \Omega \to [0, \infty)$,*

$$\int f \circ T d\mu = \int f d\mu. \tag{3}$$

Proof. *Let T be a measurable transformation and suppose equation (3) is satisfied. For any measurable set A of finite measure, then $f = \chi_A$ is an integrable function. Moreover, $f \circ T = \chi_{T^{-1}(A)}$. Note that*

$$\int f \circ T d\mu = \mu(T^{-1}(A)) \quad \text{and} \quad \int f d\mu = \mu(A),$$

it follows that $\mu(T^{-1}(A)) = \mu(A)$. Now, if A has infinite measure, write $A = \cup_{n=1}^{\infty} A_n$ since Ω is σ-finite, where A_n are pairwise disjoint and of finite measure. Then

$$\mu(T^{-1}(A)) = \mu\left(T^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = \bigcup_{n=1}^{\infty} \mu(T^{-1}(A_n)) = \bigcup_{n=1}^{\infty} \mu(A_n) = \mu(A).$$

Hence T is measure-preserving.

For the converse, if T is measure-preserving, then equation (3) holds when f is the characteristic function of a measurable set A. Now let f be an arbitrary integrable non-negative function. By The Simple Approximation Theorem[2], there is a sequence of simple functions $\{s_n\}_{n \geqslant 1}$ such that

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n \leqslant f \quad \text{and} \quad \lim_{n \to \infty} s_n = f(x) \text{ for all } x \in \Omega.$$

Note that s_n are integrable and are finite sums of characteristic functions, for each $n \geqslant 1$,

$$\int s_n \circ T d\mu = \int s_n d\mu.$$

By The Monotone Convergence Theorem[2], we conclude that

$$\int f \circ T d\mu = \lim_{n\to\infty} \int s_n \circ T d\mu = \lim_{n\to\infty} \int s_n d\mu = \int f d\mu.$$

■

Remark 1. *Of course, if T is measure-preserving on general measure space, equation (3) also holds.*

Hence, if we have a ergodic transformation τ on $(\Omega, \mathcal{F}, \mu)$. We can define a linear operator T on $L_1(\Omega, \mathcal{F}, \mu)$ by $Tf = f \circ \tau$. since τ is measure-preserving, then T satisfies:

- $f \geqslant 0$ a.e. $\Longrightarrow$ $Tf \geqslant 0$ a.e.;
- $\int_\Omega |Tf| d\mu = \int_\Omega |f| \circ \tau d\mu = \int_\Omega |f| d\mu$ by Lemma 1;
- for all $C > 0$, $|f| \leqslant C$ a.e. $\Longrightarrow$ $|Tf| \leqslant C$ a.e..

Now let us consider more general case. Given a measure space $(\Omega, \mathcal{F}, \mu)$, and a linear operator T defined in $L_1(\Omega, \mathcal{F}, \mu)$ such that

$$f \geqslant 0 \text{ a.e. } \Longrightarrow Tf \geqslant 0 \text{ a.e.} \tag{4}$$

$$\int_\Omega |Tf| d\mu \leqslant \int_\Omega |f| d\mu \tag{5}$$

$$\text{for all } C > 0,\ |f| \leqslant C \text{ a.e. } \Longrightarrow |Tf| \leqslant C \text{ a.e.} \tag{6}$$

Then the operator T defined by $Tf = f \circ \tau$ satisfies (4), (5) and (6).

Lemma 2. *If T satisfies (4), (5) and (6), then for any constant $C > 0$ and any $g \in L_1$ we have*

$$(Tg - C)^+ \leqslant T(g - C)^+. \tag{7}$$

By $(x)^+$ we mean $\max[0, x]$; we also set $(x)^- = \max[0, -x]$.

Proof. *Let*

$$g_C = \begin{cases} C, & \text{if } g > C, \\ -C, & \text{if } g < -C, \\ g, & \text{if } |g| \leqslant C, \end{cases} \qquad R_C = g - g_C.$$

It is clear that

$$R_C \leqslant (g - C)^+, \quad |g_C| \leqslant C.$$

By (4) and (6), it follows that

$$|Tg_C| \leqslant C \quad \textit{and} \quad TR_C \leqslant T(g-C)^+.$$

Thus we get

$$Tg = Tg_C + TR_C \leqslant C + T(g-C)^+,$$

hence (7) holds by the definition of $(x)^+$ and the positivity of $T(g-C)^+$. ■

Now, let us introduce some notations. Define

$$R_n(f) = \frac{f + Tf + \cdots + T^n f}{n+1} \quad \text{and} \quad R^*(f) = \sup_{n \geqslant 0} \left| \frac{f + Tf + \cdots + T^n f}{n+1} \right|.$$

Setting

$$E^n(f) = \{x : \max_{0 \leqslant m \leqslant n} (f + Tf + \cdots + T^m f) > 0\},$$
$$E(f) = \{x : \sup_{0 \leqslant n} (f + Tf + \cdots + T^n f) > 0\}.$$

It will also be convenient to introduce the function

$$\varphi_n(x_1, x_2, \cdots, x_n) = \max_{1 \leqslant m \leqslant n} (x_1 + x_2 + \cdots + x_m)^+.$$

It is clear that

$$E^n(f) = \{x : \varphi_{n+1}(f, Tf, \cdots, T^n f) > 0\}.$$

By definition, we have

$$x_1 + \varphi_n(x_2, x_3, \cdots, x_{n+1}) \geqslant \max_{1 \leqslant m \leqslant n} (x_1 + x_2 + \cdots + x_v).$$

If we assume $\varphi_n(x_1, x_2, \cdots, x_n) > 0$, then we also have

$$\varphi_n(x_1, x_2, \cdots, x_n) = \max_{1 \leqslant m \leqslant n} (x_1 + x_2 + \cdots + x_m).$$

This implies φ_n has the following property: Whenever $\varphi_n(x_1, x_2, \cdots, x_n) > 0$, then no matter what is the value of x_{n+1} we have

$$x_1 + \varphi_n(x_2, x_3, \cdots, x_{n+1}) \geqslant \varphi_n(x_1, x_2 \cdots, x_n). \tag{8}$$

Let us introduce the sets

$$E_\lambda^n(f) = \{\max_{0 \leqslant m \leqslant n} R_m(f) > \lambda\}, \quad E_\lambda(f) = \{\sup_{m \geqslant 0} R_m(f) > \lambda.\}$$

Now we are ready to show the following theorems

Theorem 1. *If T satisfies (4), (5) and (6), then for all $f \in L_p$ $(p \geqslant 1)$ and all $\lambda > 0$ we have*

(a) $\mu\{E_\lambda^n(f)\} < \infty$;

(b) $\int_{E_\lambda^n(f)} (f - \lambda) \geqslant 0$, *for all* $n \geqslant 0$.

In particular, when $f \in L_1$, we have, letting $n \to \infty$,

$$\mu\{E_\lambda(f)\} \leqslant \frac{1}{\lambda} \int_\Omega |f| d\mu.$$

Proof. *By the definition of $E_\lambda^n(f)$, we have*

$$E_\lambda^n(f) = \{x : \varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda) > 0\}.$$

When $\varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda) > 0$, then at least one of the inequalities $T^m f > \lambda$ must hold. Thus, if $f \in L_p$, we deduce that

$$\mu\{E_\lambda(f)\} \leqslant \sum_{m=0}^{n} \mu\{|T^m f| > \lambda\} \leqslant \frac{1}{\lambda^p} \sum_{m=0}^{n} \int_\Omega |T^m f|^p d\mu < \infty.$$

This shows (a).

By (8), we obtain

$$\int_{E_\lambda^n(f)} (f - \lambda) \geqslant \int_{E_\lambda^n(f)} \left\{\varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda) - \varphi_{n+1}(Tf - \lambda, \cdots, T^{n+1} f - \lambda)\right\} d\mu.$$

Thanks to Lemma 2 with $C = (m+1)\lambda$, $g = f + \cdots + T^m f$, we deduce

$$[Tf + \cdots + T^{m+1} f - (m+1)\lambda]^+ \leqslant T[f + \cdots + T^m f - (m+1)\lambda]^+$$

for any integers $m \leqslant n$. Thus we get

$$\varphi_{n+1}(Tf - \lambda, \cdots, T^n f - \lambda) \leqslant \varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda).$$

Now again by (6), this implies

$$\begin{aligned} \int_{E_\lambda^n(f)} (f - \lambda) \geqslant \int_{E_\lambda^n(f)} &[\varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda) \\ &- T\varphi_{n+1}(f - \lambda, \cdots, T^n f - \lambda)] d\mu \geqslant 0. \end{aligned}$$

That is (b), hence the proof is complete. ∎

Theorem 2. *Let X and Y be two nonnegative measurable functions and assume that $X \in L_p$ for some $p > 1$. Furthermore, suppose that for each $\lambda > 0$ we have*

(a)$\mu\{Y > \lambda\} < \infty$;

(b)$\mu\{Y > \lambda\} \leqslant \frac{1}{\lambda}\int_{Y\geqslant\lambda} X d\mu$.

Then Y must necessarily be also in L_p and

$$\int_\Omega Y^p d\mu \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega X^p d\mu.$$

Proof. *We consider $Y \in L_p$ first. We rewrite the inequality (b) in the form*

$$\lambda \int_\Omega \chi(Y,\lambda) d\mu \leqslant \int_\Omega \chi(Y,\lambda) X d\mu, \tag{9}$$

where the function $\chi(\xi,\lambda)$ is defined as

$$\chi(\xi,\lambda) = \begin{cases} 1, & \text{if } \xi > \lambda; \\ 0, & \text{if } 0 \leqslant \xi \leqslant \lambda. \end{cases} \tag{10}$$

Multiply both sides of (9) by λ^{p-2} and integrate with respect to λ from 0 to ∞. Thanks to Fubini's theorem[2], it follows that

$$\int_0^\infty \lambda^{p-1} \int_\Omega \chi(Y,\lambda) d\mu d\lambda = \int_\Omega \int_0^\infty \lambda^{p-1}\chi(Y,\lambda) d\lambda d\mu = \frac{1}{p}\int_\Omega Y^p d\mu$$

since the property of $\chi(\xi,\lambda)$ in (10). Similarly, we also get

$$\int_0^\infty \lambda^{p-2} \int_\Omega \chi(Y,\lambda) X d\mu d\lambda = \frac{1}{p-1}\int_\Omega XY^{p-1} d\mu.$$

Hence by Holder's inequality[2], we obtain

$$\frac{1}{p}\int_\Omega Y^p d\mu \leqslant \frac{1}{p-1}\int_\Omega XY^{p-1} d\mu \leqslant \frac{1}{p-1}\left[\int_\Omega X^p \mu\right]^{1/p}\left[\int_\Omega Y^p d\mu\right]^{(p-1)/p}$$

To get the result in general setting, for any given $C > 0$, the function

$$Y_C = \begin{cases} Y, & \text{if } Y < C, \\ C, & \text{if } Y \geqslant C, \end{cases}$$

satisfies also the inequality

$$\mu\{Y_C > \lambda\} \leqslant \frac{1}{\lambda}\int_{\{Y_C > \lambda\}} X d\mu.$$

That is because uf $\lambda \geqslant C$, $\mu\{Y_C > \lambda\} = 0$, and if $\lambda < C$, the set $Y_C > \lambda$ and $Y > \lambda$ are the same.

In case $\mu(\Omega) < \infty$, then $Y_C \in L_p$. By the previous argument, it follows that

$$\int_\Omega Y_C^p d\mu \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega X^p d\mu.$$

Let $C \to \infty$, we get the result for Y.

In case $\mu(\Omega) = \infty$, from the above observation at least shows that we can assume without loss that Y is bounded. Then let $0 \leqslant Y \leqslant C$ and denote $Z_\epsilon = (Y - \epsilon)^+$. By (a), we have

$$\mu\{Z_\epsilon > 0\} = \mu\{Y > \epsilon\} < \infty.$$

Moreover, since $0 \leqslant Z_\epsilon \leqslant C$, Z_ϵ must be in L_p. Now by (b), for all $\lambda > 0$ we have

$$\mu\{Z_\epsilon > \lambda\} = \mu\{Y > \epsilon + \lambda\} \leqslant \frac{1}{\lambda + \epsilon} \int_{\{Y > \lambda + \epsilon\}} X d\mu \leqslant \frac{1}{\lambda} \int_{\{Z_\epsilon > \lambda\}} X d\mu.$$

Therefore, we also show that

$$\int_\Omega Z_\epsilon^p d\mu \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega X^p d\mu.$$

Then by letting $\epsilon \to 0$, we obtain the result, thus completing the proof of the theorem. ■

By combining the results of Theorem 1 and 2 we can deduce the following

Theorem 3. *If T satisfies (4), (5) and (6) and $f \in L_p$ $(p > 1)$, then*

$$\int_\Omega [R^*(f)]^p d\mu \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega |f|^p d\mu. \tag{11}$$

Proof. *Setting*

$$R_n^*(f) = \max_{0 \leqslant m \leqslant n} \left| \frac{f + \cdots + T^m f}{m+1} \right|,$$

Theorem 1 implies that for each $\lambda > 0$,

(a) $\mu\{R_n^*(f) > \lambda\} < \infty$;

(b) $\mu\{R_n^*(f) > \lambda\} \leqslant \frac{1}{\lambda} \int_{\{R_n^*(f) > \lambda\}} |f| d\mu$.

Thus by Theorem 2, if $f \in L_p$ $(p > 1)$, we have

$$\int_\Omega [R_n^*(f)]^p d\mu \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega |f|^p d\mu.$$

By letting $n \to \infty$, we deduce the inequality in (11). ■

Let us now come back to (1) and (2). Note that $f^* = R^*(f \circ \tau)$, the hence by Lemma 1, Theorem 3 and the observations after Remark 1, it follows that

$$\|f^*\|_p = \|R^*(f \circ \tau)\|_p \leqslant \frac{p}{p-1} \|f \circ \tau\|_p = \frac{p}{p-1} \|f\|_p.$$

References

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