# Quiz ♯ 5

Due date: Thursday, April 23

**Setting**: Introduce the matrix-valued function,

$$\boldsymbol{Y}_n(z) = \begin{pmatrix} P_n(z) & \mathcal{C}(P_n w)(z) \\ cP_{n-1}(z) & c\,\mathcal{C}(P_{n-1}w)(z) \end{pmatrix}, \quad w(z) = e^{-NV(z)} \tag{1}$$

where

$$\mathcal{C}(P_n w)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_n(u)w(u)du}{u-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{2}$$

and the constant,

$$c = -\frac{2\pi i}{h_{n-1}}, \tag{3}$$

is chosen in such a way that

$$\mathcal{C}(P_n w)(z) \sim \frac{1}{z^n} + \cdots . \tag{4}$$

The monic orthogonal polynomials $P_n(z) = z^n + \cdots$ satisfy the orthogonality condition,

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)e^{-NV(x)}dx = h_n\delta_{mn}, \tag{5}$$

and the three term recurrence relation,

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x), \tag{6}$$

$$\gamma_n = \left(\frac{h_n}{h_{n-1}}\right)^{1/2} > 0, \quad n \geqslant 1; \quad \gamma_0 = 0. \tag{7}$$

The $\psi$-functions are defined as

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} P_n(x)e^{-NV(x)/2}. \tag{8}$$

The function $\boldsymbol{Y}_n$ solves the following Riemann-Hilbert Problem (RHP):

1. $\boldsymbol{Y}_n(z)$ is analytic on $\mathbb{C}^+ \equiv \{\mathfrak{Im} z \geqslant 0\}$ and $\mathbb{C}^- \equiv \{\mathfrak{Im} z \leqslant 0\}$ and is two-valued on $\mathbb{R} = \mathbb{C}^+ \cap \mathbb{C}^-$.
2. For any real $x$,

$$\boldsymbol{Y}_{n+}(x) = \boldsymbol{Y}_{n-}(x)J_Y(x), \quad J_Y(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \tag{9}$$

3. As $z \to \infty$,

$$\boldsymbol{Y}_n(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\boldsymbol{Y}_k}{z^k}\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{10}$$

   where $\boldsymbol{Y}_k$, $k = 1, 2, \cdots$, are some constant $2 \times 2$ matrices.

**Problem 1.** Prove that

$$\gamma_n^2 = [\boldsymbol{Y}_1]_{21}[\boldsymbol{Y}_1]_{12}, \tag{11}$$

$$\beta_{n-1} = \frac{[\boldsymbol{Y}_2]_{21}}{[\boldsymbol{Y}_1]_{21}} - [\boldsymbol{Y}_1]_{11}, \tag{12}$$

and

$$K_N(x,y) \equiv \gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y} \tag{13}$$

$$= -e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{Y}_{N+}^{-1}(y)\mathbf{Y}_{N+}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

*Proof.* Introduce the notation that

$$P_n(z) = z^n + p_{n,n-1}z^{n-1} + p_{n,n-2}z^{n-2} + \cdots + p_{n,1}z + p_{n,0},$$

then we have

$$p_{n,n-1} = p_{n+1,n} + \beta_n \tag{15}$$

by comparing the coefficient of $x^n$ in (6). From (10), we have

$$Y_n(z) \sim \begin{pmatrix} z^n + [\mathbf{Y}_1]_{11}z^{n-1} + [\mathbf{Y}_2]_{11}z^{n-2} + \cdots & [\mathbf{Y}_1]_{12}z^{-n-1} + [\mathbf{Y}_2]_{12}z^{-n-2} + \cdots \\ [\mathbf{Y}_1]_{21}z^{n-1} + [\mathbf{Y}_2]_{21}z^{n-2} + \cdots & z^{-n} + [\mathbf{Y}_1]_{22}z^{-n-1} + [\mathbf{Y}_2]_{22}z^{-n-2} + \cdots \end{pmatrix}.$$

Compare this with (1), it follows that

$$[\mathbf{Y}_1]_{11} = p_{n,n-1}, [\mathbf{Y}_1]_{12} = -\frac{h_n}{2\pi i}, [\mathbf{Y}_1]_{21} = c, [\mathbf{Y}_2]_{21} = cp_{n-1,n-2}.$$

Thus we conclude

$$[\mathbf{Y}_1]_{21}[\mathbf{Y}_1]_{12} = c \times \left(-\frac{h_n}{2\pi i}\right) = \frac{h_n}{h_{n-1}} = \gamma_n^2$$

and

$$\frac{[\mathbf{Y}_2]_{21}}{[\mathbf{Y}_1]_{21}} - [\mathbf{Y}_1]_{11} = p_{n-1,n-2} - p_{n,n-1} = \beta_{n-1}$$

by (7) and (15).

As for $K_N(x,y)$, we have

$$\begin{aligned}
&\gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y} \\
=&\frac{\gamma_N}{x-y}\left(\frac{1}{\sqrt{h_N h_{N-1}}} P_N(x)e^{-\frac{NV(x)}{2}} P_{N-1}(y)e^{-\frac{NV(y)}{2}} - \frac{1}{\sqrt{h_N h_{N-1}}} P_{N-1}(x)e^{-\frac{NV(x)}{2}} P_N(y)e^{-\frac{NV(y)}{2}}\right) \\
=&\frac{\gamma_N}{\sqrt{h_N h_{N-1}}(x-y)} e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} (P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)).
\end{aligned}$$

On the other hand, we also have

$$\mathbf{Y}_{N+}^{-1}(y) = \begin{pmatrix} c\,\mathcal{C}(P_{N-1}w)(z) & -\mathcal{C}(P_N w)(z) \\ -cP_{N-1}(z) & P_N(z) \end{pmatrix}$$

since $\det \boldsymbol{Y}_N(z) \equiv 1$ by Problem 2. Hence

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{Y}_{N+}^{-1}(y)\mathbf{Y}_{N+}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -cP_{N-1}(y) & P_N(y) \end{pmatrix} \begin{pmatrix} P_N(x) \\ cP_{N-1}(x) \end{pmatrix} = c(P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)).$$

Thus

$$\begin{aligned}
&- e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \boldsymbol{Y}_{N+}^{-1}(y)\boldsymbol{Y}_{N+}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
=& - e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} \frac{1}{2\pi i(x-y)} c(P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)) \\
=& e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} \frac{1}{h_N(x-y)} (P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)) \\
=& \frac{\gamma_N}{\sqrt{h_N h_{N-1}}(x-y)} e^{-\frac{NV(x)}{2}} e^{-\frac{NV(y)}{2}} (P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)) \\
=& \gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y} = K_N(x,y).
\end{aligned}$$

□

**Problem 2.** Prove that

$$\det \boldsymbol{Y}_n(z) \equiv 1. \tag{16}$$

*Proof.* By definition, it suffices to show

$$1 = \frac{c}{2\pi i} \int_{-\infty}^{\infty} \frac{(P_{n-1}(u)P_n(z) - P_n(u)P_{n-1}(z))w(u)}{u-z} du.$$

We can spli it into two parts, that is

$$\frac{c}{2\pi i} \int_{-\infty}^{\infty} P_{n-1}(u) \frac{P_n(z) - P_n(u)}{u-z} w(u) du$$

and

$$\frac{c}{2\pi i} \int_{-\infty}^{\infty} P_n(u) \frac{P_{n-1}(u) - P_{n-1}(z)}{u-z} w(u) du.$$

Note that $\frac{P_n(u) - P_n(z)}{u-z}$ is a monic polynomial with degree $n-1$, it follows that

$$\frac{c}{2\pi i} \int_{-\infty}^{\infty} P_{n-1}(u) \frac{P_n(z) - P_n(u)}{u-z} w(u) du = -\frac{c}{2\pi i} h_{n-1} \tag{17}$$

by definition of $P_n(z)$. Similarly, we obtain

$$\frac{c}{2\pi i} \int_{-\infty}^{\infty} P_n(u) \frac{P_{n-1}(u) - P_{n-1}(z)}{u-z} w(u) du = 0. \tag{18}$$

Combine (3), (17) with (18), we have

$$\frac{c}{2\pi i} \int_{-\infty}^{\infty} \frac{(P_{n-1}(u)P_n(z) - P_n(u)P_{n-1}(z))w(u)}{u-z} du = -\frac{c}{2\pi i} h_{n-1} = 1.$$

□

**Bonus Problem.** Prove that the RHP (9), (10) has a unique solution.

*Proof.* We have already seen that the RHP (9), (10) does have a solution. Suppose $\boldsymbol{W}_n(z)$ is another solution of the RHP (9), (10). Let us consider $\boldsymbol{\Phi}(z) = \boldsymbol{W}_n(z)\boldsymbol{Y}_n(z)^{-1}$, where $\boldsymbol{Y}_n(z)^{-1}$ exists by Problem 2. Then we have

$$\begin{aligned}\boldsymbol{\Phi}_{n+}(z) =&\boldsymbol{W}_{n+}(z)\boldsymbol{Y}_{n+}^{-1}(z) = (\boldsymbol{W}_{n-}(z)J_Y(z))\left(\boldsymbol{Y}_{n-}(z)J_Y(z)\right)^{-1}\\ =&\boldsymbol{W}_{n-}(z)J_Y(z)(J_Y(z))^{-1}(\boldsymbol{Y}_{n-}(z))^{-1} = \boldsymbol{W}_{n-}(z)(\boldsymbol{Y}_{n-}(z))^{-1} = \boldsymbol{\Phi}_{n-}(z).\end{aligned}$$

This implies $\boldsymbol{\Phi}(z)$ is holomorphic in the complex plane.

On the other hand, we have

$$\boldsymbol{W}_n(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\boldsymbol{W}_k}{z^k}\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}, \quad z \to \infty$$

and

$$\boldsymbol{Y}_n^{-1}(z) \sim \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix}\left(I + \sum_{k=1}^{\infty} \frac{\boldsymbol{Y}_k}{z^k}\right)^{-1} \sim \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix}\left(I - \sum_{k=1}^{\infty} \frac{\tilde{\boldsymbol{Y}}_k}{z^k}\right), \quad z \to \infty$$

where $\tilde{\boldsymbol{Y}}_1 = \boldsymbol{Y}_1$ and $\boldsymbol{Y}_k$, $\boldsymbol{W}_k$, $k = 1, 2, \cdots$, are some constant $2 \times 2$ matrices. Thus

$$\boldsymbol{\Phi}(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\boldsymbol{W}_k}{z^k}\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix}\left(I - \sum_{k=1}^{\infty} \frac{\tilde{\boldsymbol{Y}}_k}{z^k}\right) \sim I + \sum_{k=1}^{\infty} \frac{\boldsymbol{\lambda}_k}{z^k}, \quad z \to \infty$$

where $\boldsymbol{\lambda}_k$, $k = 1, 2, \cdots$, are some constant $2 \times 2$ matrices. Combine this with previous arguments, we deduce that $\boldsymbol{\Phi}(z)$ is a bounded entire function, hence $\boldsymbol{\Phi}(z)$ is constant. By the asymptotic of $\boldsymbol{\Phi}(z)$ at infinity, it follows that $\boldsymbol{\Phi}(z) = I$. Therefore $\boldsymbol{W}_n(z)\boldsymbol{Y}_n(z)^{-1} = I$, which in turn implies $\boldsymbol{W}_n(z) = \boldsymbol{Y}_n(z)$. Thus the RHP (9), (10) has a unique solution. □