Quiz ♯ 4

Due date: Thursday, April 2

Setting: Let $\mu = \lim_{n\to\infty} \mu_n$ be a limiting Gibbs measure of the one-dimensional Ising model with zero magnetic field on the lattice $\mathbb{Z}$. Respectively, a configuration σ for μ is

$$\sigma = \{\sigma(x) = \pm 1 \mid x \in \mathbb{Z}\}.$$

For a random variable $f(\sigma)$ we denote

$$\langle f(\sigma)\rangle = \int f(\sigma) d\mu(\sigma).$$

Let

$$\sigma_n(k) = \frac{1}{\sqrt{n}} \sum_{(k-1)n+1 \leqslant x \leqslant kn} \sigma x, \quad k \in \mathbb{Z},$$

be normed block spins.

1. Calculate the limiting two-point normed block spin correlation function,

$$K_2(k_1, k_2) = \lim_{n\to\infty} \langle \sigma_n(k_1)\sigma_n(k_2)\rangle,$$

in two cases: (1) $k_1 = k_2$ and (2) $k_1 < k_2$.

Solution. *Note that we have*

$$K_2(k_1, k_2) = \lim_{n\to\infty} \langle \sigma_n(k_1)\sigma_n(k_2)\rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{x\in V_n(k_1), y\in V_n(k_2)} C(x, y),$$

where $V_n(k) = [k(n-1)+1, kn] \cap \mathbb{Z}$ *and* $C(x, y)$ *is the two point correlation function. As we consider the one-dimensional Ising model with zero magnetic field on the lattice* $\mathbb{Z}$*, we have*

$$C(x, y) = q^{|x-y|}, \quad q = \tan(\beta J).$$

If $k_1 = k_2$*, we have*

$$\sum_{x\in V_n(k_1), y\in V_n(k_2)} C(x, y) = n + 2(n-1)q + 2(n-2)q^2 + \cdots + 2q^{n-1}.$$

Thus

$$K_2(k_1, k_2) = 1 + 2q + 2q^2 + \cdots = \frac{1+q}{1-q} = \mathcal{D}.$$

If $k_1 < k_2 - 1$*, then all the terms in* $K_2(k_1, k_2)$ *have the form* $p(n)q^{kn}$*, where* p *is a polynomial and* $k \in Z$*. Since* $|q| < 1$*, we have*

$$p(n)q^{kn} \to 0, \quad \text{as } n \to \infty.$$

Thus $K_2(k_1, k_2) = 0$ *for this case.*

If $k_1 = k_2 - 1$, *then we have*

$$\sum_{x \in V_n(k_1), y \in V_n(k_2)} C(x, y) = q + 2q^2 + \cdots + nq^n + \cdots + q^{2n-1},$$

henceThus

$$K_2(k_1, k_2) = \lim_{n \to \infty} \frac{1}{n}(q + 2q^2 + \cdots + nq^n + \cdots + q^{2n-1}) = 0.$$

By above, we have

$$K(k_1, k_2) = \begin{cases} \mathcal{D}, & k_1 = k_2; \\ 0, & k_1 < k_2, \end{cases}$$

where $\mathcal{D} = \dfrac{1+q}{1-q}$ *and* $q = \tan(\beta J)$. ■

2. Calculate the limiting four-point normed block spin correlation function,

$$K_4(k_1, k_2, k_3, k_4) = \lim_{n \to \infty} \langle \sigma_n(k_1)\sigma_n(k_2)\sigma_n(k_3)\sigma_n(k_4) \rangle,$$

in two cases: (1) $k_1 = k_2 = k_3 = k_4$ and (2) $k_1 < k_2 < k_3 < k_4$.

Solution. *By Wick's Formula, we have*

$$K_4(k_1, k_2, k_3, k_4) = K_2(k_1, k_2)K_2(k_3, k_4) + K_2(k_1, k_3)K_2(k_2, k_4) + K_2(k_1, k_4)K_2(k_2, k_3).$$

If $k_1 = k_2 = k_3 = k_4$, *then*

$$K_4(k_1, k_2, k_3, k_4) = 3\mathcal{D}^2$$

by Problem 1.

If $k_1 < k_2 < k_3 < k_4$, *then*

$$K_4(k_1, k_2, k_3, k_4) = 0$$

by Problem 1. ■

3. Calculate the limiting normed block spin correlation function,

$$K_{2m} = \lim_{n \to \infty} \langle [\sigma_n(k)]^{2m} \rangle, \quad m = 1, 2, \cdots.$$

Solution. *Again by Wick's Formula, we have*

$$K_{2m}(k_1, \cdots, k_{2m}) = \sum_{\pi \in \mathcal{P}_2(2m)} \prod_{(i,j) \in \pi} K_2(k_i, k_j),$$

where $\mathcal{P}_2$ *denotes the set of all pairings of the set* $\{1, 2, \cdots, 2m\}$.

Since $k_1 = k_2 = \cdots = k_{2m}$*, we conclude*

$$K_{2m} = \#\mathcal{P}_2(2m)\mathcal{D}^m.$$

In fact, we can calculate it and get $\#\mathcal{P}_2(2m) = (2m-1)!!$*, where*

$$(2m-1)!! = (2m-1)\cdot(2m-3)\cdot\cdots\cdot 3\cdot 1.$$

Hence

$$K_{2m} = (2m-1)!!\mathcal{D}^m = \frac{1}{\sqrt{2\pi\mathcal{D}}}\int_{\mathbb{R}} x^{2m} e^{-\frac{x^2}{2\mathcal{D}}}\,dx.$$

The last formula can be proved by induction and integration by parts. ■