Quiz ♯ 2

Due date: Thursday, February 5

1. Calculate the two-point correlation function

$$K_2(x_1, x_2) = \lim_{n\to\infty} \sum_{\sigma} \sigma(x_1)\sigma(x_2)\mu_n^0(\sigma)$$

in the Ising model on the Bethe lattice with zero magnetic field in the disordered phase.

Solution. *Denote a path from the origin to boundary of B_n that passes x_1 and x_2 by A_n. Inductively, we can get a sequence of pathes with increasing length such that $A_n \subset A_{n+1}$.*

For brevity, here A means A_n, then

$$\begin{aligned}
\mu_n^0(\sigma)\big|_A &= \sum_{\sigma'\in\Sigma,\sigma'|_A=\sigma|_A} \frac{1}{Z_n} e^{\beta J \sum_{<x,y>\in L_n} \sigma'(x)\sigma'(y)} \\
&= \sum_{\sigma'\in\Sigma,\sigma'|_A=\sigma|_A} \frac{1}{Z_n} e^{\beta J \sum_{<x,y>\in A} \sigma'(x)\sigma'(y) + \beta J \sum_{<x,y>\in L_n\setminus A} \sigma'(x)\sigma'(y) + \beta J \sum_{<x,y>\in L_n, x\in A, y\in\partial A} \sigma'(x)\sigma'(y)} \\
&= e^{\beta J \sum_{<x,y>\in A} \sigma(x)\sigma(y)} \sum_{\sigma'\in\Sigma,\sigma'|_A=\sigma|_A} \frac{1}{Z_n} e^{\beta J \sum_{<x,y>\in L_n\setminus A} \sigma'(x)\sigma'(y) + \beta J \sum_{<x,y>\in L_n, x\in A, y\in\partial A} \sigma'(x)\sigma'(y)}
\end{aligned}$$

It is easy to see that

$$\sum_{\sigma'\in\Sigma,\sigma'|_A=\sigma|_A} \frac{1}{Z_n} e^{\beta J \sum_{<x,y>\in L_n\setminus A} \sigma'(x)\sigma'(y) + \beta J \sum_{<x,y>\in L_n, x\in A, y\in\partial A} \sigma'(x)\sigma'(y)}$$

is independent of $\sigma|_A$. Hence, we can denote it by $1/Z_A$ (it depends on n). Consequently, we obtain

$$\mu_n^0(\sigma)\big|_A = \frac{1}{Z_A} e^{\beta J \sum_{<x,y>\in A} \sigma(x)\sigma(y)}.$$

Hence the problem is equivalent to the 1D Ising model with zero magnetic field and free boundary conditions.

Now, use the same argument of Problem 1 of Quiz 1, we conclude

$$K_2(x_1, x_2) = \lim_{n\to\infty} \sum_{\sigma} \sigma(x_1)\sigma(x_2)\mu_n^0(\sigma) = [\tanh(\beta J)]^{d(x_1,x_2)}.$$

■

2. Calculate the two-point correlation function

$$K_2(x_1, x_2) = \lim_{n\to\infty} \sum_{\sigma} \sigma(x_1)\sigma(x_2)\mu_n^+(\sigma)$$

in the Ising model on the Bethe lattice with zero magnetic field in the $(+)$-phase.

Solution. *Similar to Problem 1, we denote a path from the origin to boundary of B_n that passes x_1 and x_2 by A_n. Inductively, we can get a sequence of pathes with increasing length such that $A_n \subset A_{n+1}$. By similar arguments in Problem 1 and compatible conditions, we obtain*

$$\mu_n^+(\sigma)\big|_{A_n} = \frac{1}{Z_{A_n}} e^{\beta J \sum_{<x,y>\in A_n} \sigma(x)\sigma(y) + x^* \sum_{x\in A_n} \sigma(x)}.$$

Hence, we can reduce this problem to 1D Ising model with magnetic field. Hence, we have

$$K_2(x_1, x_2) = s_{11} + s_{12}s_{21}q^{d(x_1,x_2)},$$

where

$$q = \frac{\lambda_2}{\lambda_1}, s_{11} = (Se_1, e_1), s_{12} = (Se_2, e_1), s_{21} = (Se_1, s_2),$$

and

$$\lambda_{1,2} = e^{\beta J}\cosh x^* \pm \sqrt{e^{2\beta J}\sinh^2 x^* + e^{-2\beta J}}, S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

e_1 and e_2 are the eigenvectors (with special properties) of T corresponding to λ_1 and λ_2, respectively. ■

Remark. *I found the formula below on my notes*

$$\mu(\sigma, h)|_A = \frac{1}{Z_A} e^{\beta J \sum_{<x,y>\in L_A} \sigma(x)\sigma(y) + \sum_{x\in\partial A} h\sigma(x)},$$

where A is a path and h is a constant. However, here $\mu(\sigma, h)|_A$ depends on $\sum_{x\in\partial A} h\sigma(x)$.

Using the similar method in Problem 1, I got

$$\mu^+(\sigma)|_A = \frac{1}{Z_A} \sum_{\sigma|_A \text{is fixed}} e^{\beta J \sum_{<x,y>\in L_A} \sigma(x)\sigma(y) + \beta J \sum_{x\in A, y\in\partial A} \sigma(x)\sigma(y) + \sum_{x\in\partial A} x^*\sigma(x)}.$$

To get this formula, I use the similar method in Bonus Problem and the definition of x^* such that

$$\left(e^{\beta J\sigma(x) + x^*} + e^{-\beta J\sigma(x) - x^*}\right)^k = e^{x^*\sigma(x)}$$

where k is the degree of the graph. If I choose similar path A_n as problem 1. Then I have

$$\mu^+(\sigma)|_A = \frac{1}{Z_A} e^{\beta J \sum_{<x,y>\in L_A} \sigma(x)\sigma(y)} \sum_{\sigma|_A \text{is fixed}} e^{\beta J \sum_{x\in A, y\in\partial A} \sigma(x)\sigma(y) + \sum_{x\in\partial A} x^*\sigma(x)}.$$

Denote the origin of A_n by ζ_0 and the end on B_n by ζ_n, then by the definition of x^*, we get

$$\sum_{\sigma|_A \text{is fixed}} e^{\beta J \sum_{x\in A, y\in\partial A} \sigma(x)\sigma(y) + \sum_{x\in\partial A} x^*\sigma(x)} = e^{(1-1/k)x^* \sum_{x\in A} \sigma(x) + 1/k\sigma(\zeta_n)}.$$

Hence the problem reduces to the 1D Ising model with magnetic field and boundary condition. Then we can use the solution part(some modifications such as changing x^* to $(1-1/k)x^*$). ■

3. Calculate the free energy

$$F = -\frac{1}{\beta} \lim_{n\to\infty} \frac{1}{|B_n|} \ln Z_n, \quad Z_n = \sum_{\sigma} e^{\beta J \sum_{<x,y>\in L_n} \sigma(x)\sigma(y)},$$

in the Ising model on the Bethe lattice with zero magnetic field with free boundary conditions.

Solution. *Let us consider the relation between Z_{n+1} and Z_n. In fact*

$$\begin{aligned} Z_{n+1} &= \sum_{\sigma} e^{\beta J \sum_{<x,y>\in L_{n+1}} \sigma(x)\sigma(y)} \\ &= \sum_{\sigma} e^{\beta J \sum_{<x,y>\in L_n} \sigma(x)\sigma(y) + \beta J \sum_{<x,y>\in L_n, x\in S_n, y\in S_{n+1}} \sigma(x)\sigma(y)}. \end{aligned}$$

If we fix $\sigma|_{V_n}$, then

$$\sum_{\sigma} e^{\beta J \sum_{<x,y>\in L_n, x\in S_n, y\in S_{n+1}} \sigma(x)\sigma(y)} = \prod_{x\in S_{n+1}} (2\cosh(\beta J)).$$

Thus

$$Z_{n+1} = Z_n \prod_{x\in S_{n+1}} (2\cosh(\beta J)).$$

By induction, it is easy to show that

$$Z_n = 2(2\cosh(\beta J))^{|B_n|}.$$

It follows that

$$F = -\frac{1}{\beta} \lim_{n\to\infty} \frac{1}{|B_n|} \ln Z_n = -\frac{1}{\beta} \ln(2\cosh(\beta J)).$$

■