Quiz ♯ 1

Due date: Thursday, January 29

1. Calculate the four-point correlation function

$$K_4(x_1,x_2,x_3,x_4)=\lim_{n\to\infty}\sum_{\sigma}\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\mu_n(\sigma)$$

in the 1D Ising model with zero magnetic field.

Solution. *For $h=0$, we can get the eigenvalues of transfer matrix are*

$$\lambda_{1,2}=e^{\beta J}\pm e^{-\beta J},$$

and the corresponding eigenvectors are

$$e_1=\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix},\quad e_2=\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}.$$

Therefore, we can obtain that

$$Se_1=e_2,\quad Se_2=e_1,$$

where $S=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Denote

$$Z_n(x_1,x_2,x_3,x_4)=\sum_{\sigma}\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)e^{-\beta H_n(\sigma)},$$

then we will have

$$Z_n(x_1,x_2,x_3,x_4)=(t_1,T^{x_1-1}ST^{x_2}ST^{x_3}ST^{x_4}ST^{n-x_4}t_2). \tag{1}$$

If we set

$$t_1=\alpha_1e_1+\beta_1e_2,\quad t_2=\alpha_2e_1+\beta_2e_2,$$

then by (1), one has

$$\begin{aligned}Z_n(x_1,x_2,x_3,x_4)=&(t_1,\alpha_2\lambda_1^{n-x_4+x_3-x_2+x_1-1}\lambda_2^{x_4-x_3+x_2-x_1}e_1)\\&+(t_1,\beta_2\lambda_2^{n-x_4+x_3-x_2+x_1-1}\lambda_1^{x_4-x_3+x_2-x_1}e_2)\\=&\alpha_1\alpha_2\lambda_1^{n-x_4+x_3-x_2+x_1-1}\lambda_2^{x_4-x_3+x_2-x_1}\\&+\beta_1\beta_2\lambda_2^{n-x_4+x_3-x_2+x_1-1}\lambda_1^{x_4-x_3+x_2-x_1}\end{aligned}$$

since $(e_1,e_1)=1$, $(e_2,e_2)=1$ and $(e_1,e_2)=0$. Moreover, it is trivial that

$$\alpha_1=(t_1,e_1)>0,\quad \alpha_2=(t_2,e_1)>0.$$

Now, note that $\lambda_1 > |\lambda_2|$, we can get

$$\begin{aligned} K_4(x_1,x_2,x_3,x_4) &= \lim_{n\to\infty} \frac{Z_n(x_1,x_2,x_3,x_4)}{Z_n} \\ &= \lim_{n\to\infty} \frac{\alpha_1\alpha_2\lambda_1^{n-x_4+x_3-x_2+x_1-1}\lambda_2^{x_4-x_3+x_2-x_1}+\cdots}{\alpha_1\alpha_2\lambda_1^{n-1}+\cdots} \\ &= \left(\frac{\lambda_2}{\lambda_1}\right)^{x_4-x_3+x_2-x_1} \\ &= [\tanh(\beta J)]^{x_4-x_3+x_2-x_1}. \end{aligned}$$

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2. Calculate the free energy and the two-point correlation function $K_2(x_1, x_2)$ in the 1D Potts model with 3 states and zero magnetic field.

Solution. *I will use the notation in '1D Potts model with external magnetic field'. Since $h = 0$, we have*

$$t_1 = \begin{pmatrix} e^{\beta J(b_1,\omega_0)} \\ e^{\beta J(b_1,\omega_1)} \\ e^{\beta J(b_1,\omega_2)} \end{pmatrix}, \quad t_2 = \begin{pmatrix} e^{\beta J(b_2,\omega_0)} \\ e^{\beta J(b_2,\omega_1)} \\ e^{\beta J(b_2,\omega_2)} \end{pmatrix},$$

and the transfer matrix

$$T = \begin{pmatrix} e^{\beta J(\omega_0,\omega_0)} & e^{\beta J(\omega_0,\omega_1)} & e^{\beta J(\omega_0,\omega_2)} \\ e^{\beta J(\omega_1,\omega_0)} & e^{\beta J(\omega_1,\omega_1)} & e^{\beta J(\omega_1,\omega_2)} \\ e^{\beta J(\omega_2,\omega_0)} & e^{\beta J(\omega_2,\omega_1)} & e^{\beta J(\omega_2,\omega_2)} \end{pmatrix} = \begin{pmatrix} e^{\beta J} & e^{-\frac{1}{2}\beta J} & e^{-\frac{1}{2}\beta J} \\ e^{-\frac{1}{2}\beta J} & e^{\beta J} & e^{-\frac{1}{2}\beta J} \\ e^{-\frac{1}{2}\beta J} & e^{-\frac{1}{2}\beta J} & e^{\beta J} \end{pmatrix}.$$

Then, we will have

$$Z_n = (t_1, T^{n-1}t_2). \tag{2}$$

On the other hand, the eigenvalues of T are $\lambda_1 = e^{\beta J} + 2e^{-\frac{1}{2}\beta J}$ and $\lambda_2 = \lambda_3 = e^{\beta J} - e^{-\frac{1}{2}\beta J}$. Let e_1, e_2 and e_3 be the corresponding eigenvectors respectively such that they are an orthonormal basis of $\mathbb{R}^3$ and e_1 has positive components. Moreover, denote

$$t_1 = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \quad t_2 = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3.$$

In fact $e_1 = (\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$, hence $\alpha_1 = (t_1, e_1) > 0$ and $\alpha_2 = (t_2, e_1) > 0$. Then by (2), we obtain

$$Z_n = \alpha_1\alpha_2\lambda_1^{n-1} + \beta_1\beta_2\lambda_2^{n-1} + \gamma_1\gamma_2\lambda_3^{n-1}. \tag{3}$$

Note that $\lambda_1 > |\lambda_2| = |\lambda_3|$, it follows that

$$F = -\lim_{n\to\infty} \frac{1}{n} \ln Z_n = -\ln\lambda_1 = -\ln(e^{\beta J} + 2e^{-\frac{1}{2}\beta J}).$$

Now, let us consider the two-point correlation function $K_2(x_1, x_2)$, which should be defined as

$$K_2(x_1, x_2) = \sum_{\sigma} (\sigma(x_1), \sigma(x_2))\mu(\sigma).$$

Note that

$$K_2(x_1, x_2) = \sum_{\sigma} \sigma(x_1)^{(1)}\sigma(x_2)^{(1)}\mu(\sigma) + \sum_{\sigma} \sigma(x_1)^{(2)}\sigma(x_2)^{(2)}\mu(\sigma).$$

If we denote

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Then, we have

$$K_2(x_1, x_2) = \lim_{n\to\infty} \frac{(t_1, T^{x_1-1}S_1T^{x_2-x_1}S_1T^{n-x_2}t_2) + (t_1, T^{x_1-1}S_2T^{x_2-x_1}S_2T^{n-x_2}t_2)}{Z_n}. \tag{4}$$

Actually, we can choose $e_2 = (\sqrt{2}/\sqrt{3}, -1/\sqrt{6}, -1/\sqrt{6})^T$ and $e_3 = (0, \sqrt{2}/2, -\sqrt{2}/2)^T$. From direct calculation, we can get

$$S_1e_1 = \frac{\sqrt{2}}{2}e_2, S_1e_2 = \frac{\sqrt{2}}{2}e_1 + \frac{1}{2}e_2, S_1e_3 = -\frac{1}{2}e_3,$$

and

$$S_2e_1 = \frac{\sqrt{3}}{2}e_3, S_2e_2 = -\frac{1}{2}e_3, S_2e_3 = \frac{\sqrt{2}}{2}e_1 - \frac{1}{2}e_2.$$

Now it is easy to see,

$$(t_1, T^{x_1-1}S_1T^{x_2-x_1}S_1T^{n-x_2}t_2) = \frac{1}{2}\alpha_1\alpha_2\lambda_1^{n-x_2+x_1-1}\lambda_2^{x_2-x_1} + \frac{\sqrt{2}}{4}\alpha_2\beta_1\lambda_1^{n-x_2}\lambda_2^{x_2-1} + \cdots \tag{5}$$

and

$$(t_1, T^{x_1-1}S_2T^{x_2-x_1}S_2T^{n-x_2}t_2) = \frac{1}{2}\alpha_1\alpha_2\lambda_1^{n-x_2+x_1-1}\lambda_2^{x_2-x_1} - \frac{\sqrt{2}}{4}\alpha_2\beta_1\lambda_1^{n-x_2}\lambda_2^{x_2-1} + \cdots. \tag{6}$$

Consequently, by (3), (4), (5) and (6), we conclude that

$$K_2(x_1, x_2) = \left(\frac{\lambda_2}{\lambda_1}\right)^{x_2-x_1} = \left(\frac{e^{\beta J} - e^{-\frac{1}{2}\beta J}}{e^{\beta J} + 2e^{-\frac{1}{2}\beta J}}\right)^{x_2-x_1}.$$

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3. Calculate the free energy in the classical 1D XYZ-model.

Solution. *We will use the spherical coordinate on $\mathbb{S}^2$. Then it is similar to the classical XY-model. We have*

$$Z_n = \int_{(\mathbb{S}^2)^n} t_1(\sigma_{-n_1}) \prod_{j=n_1}^{n_2-1} t(\sigma_j, \sigma_{j+1})t_2(\sigma_{n_2})d\sigma_{-n_1}\cdots d\sigma_{n_2}.$$

Consider the transfer operator

$$Tf(\theta_1,\phi_1) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} K(\theta_1\phi_1,\theta_2\phi_2)f(\theta_2,\phi_2)d\theta_2 d\phi_2, \tag{7}$$

where

$$K(\theta_1\phi_1,\theta_2\phi_2) = \exp(K\cos\Theta), \tag{8}$$

$$\cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)$$

and $K = \beta J$. *Then* $Z_n = (t_1, T^n t_2)$. *In particular, for classical XYZ-model, we have* $Z_n = (1, T^n 1)$.

We are going to find the eigenvalues and eigenfunctions of T. *Note that the kernel (8) is real and symmetric and is therefore of the Hilbert-Schmidt type. In this case, it can be shown that* T *possesses a complete set of mutually orthogonal eigenfunctions and that all eigenvalues are real.*

The correct set of eigenfunctions of T *are the spherical harmonics* $(4\pi)^{1/2}Y_l^m(\theta,\phi)$, *which can be expressed in terms of associated Legendre functions as follows:*

$$Y_l^m(\theta,\phi) = (-1)^m\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\exp(im\phi), \tag{9}$$

with

$$Y_l^{-m}(\theta,\phi) = (-1)^m Y_l^{m*}(\theta,\phi).$$

To verify this statement, we evaluate the right-hand side of (7) using the expansion

$$\exp(K\cos\Theta) = \left(\frac{\pi}{2K}\right)^{1/2}\sum_{l=0}^{\infty}(2l+1)I_{l+\frac{1}{2}}(K)P_l(\cos\Theta)$$

(where $I_{l+\frac{1}{2}}(x)$ *are modified Bessel functions of the first kind) and the addition theorem for spherical harmonics*

$$P_l(\cos\Theta) = 4\pi(2l+1)^{-1}\sum_{m=-l}^{l} Y_l^{m*}(\theta_2,\phi_2)Y_l^m(\theta_1,\phi_1).$$

The integrations over (θ_2,ϕ_2) *can now be easily performed using the standard result*

$$\int_0^{2\pi}\int_0^{\pi} Y_l^{m*}(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)d\theta d\phi = \delta_{ll'}\delta_{mm'}.$$

It is found that $(4\pi)^{1/2}Y_l^m(\theta,\phi)$ *is an eigenfunction of* T *with a corresponding eigenvalue*

$$\lambda_l(K) = \left(\frac{\pi}{2K}\right)^{1/2} I_{l+\frac{1}{2}}(K).$$

For free energy, we just need find the largest eigenvalue of T, *which is*

$$\lambda_0 = \left(\frac{\pi}{2K}\right)^{1/2} I_{\frac{1}{2}}(K) = \left(\frac{\pi}{2K}\right)^{1/2}\left(\frac{2}{\pi K}\right)^{1/2}\sinh K = \frac{\sinh K}{K}.$$

Hence the free energy is $-\frac{1}{\beta}\ln\frac{\sinh(\beta J)}{\beta J}$. ■