Homework Assignment ♯ 2

Due date: Wednesday, December 15

1. Let X be a vector field and L_X be its Lie Derivative whose action on an arbitrary tensor field T is defined by the equation,

$$L_X T := \frac{d}{dt}\Phi_t^* T\Big|_{t=0}, \tag{1}$$

where Φ is the phase flow of the field X. Let $\omega = \sum_j \omega_j dx^j$ and $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$ be a one-form an a vector field, respectively. Prove that,

$$L_X \omega = \sum_j \left(\sum_k \left(X^k \frac{\partial \omega_j}{\partial x^k} + \omega_k \frac{\partial X^k}{\partial x^j} \right) \right) dx^j \tag{2}$$

and

$$L_X Y = \sum_j \left(\sum_k \left(Y^k \frac{\partial X^j}{\partial x^k} - X^k \frac{\partial Y^j}{\partial x^k} \right) \right) \frac{\partial}{\partial x^j} \equiv [Y, X]. \tag{3}$$

Remark. *By convention, in case of vector field, $\Phi_t^* Y \equiv (\Phi_t)_* Y$.*

Proof. *Note that the right-hand sides of (1) and (2) are linear with respect to ω, so we just need consider the sepecial case that $\omega = f dx^j$. Let (U, φ) be the correspondent local coordinate system, then*

$$L_X(f dx^j) = \frac{d}{dt}\Phi_t^*(f dx^j)\Big|_{t=0} = \lim_{t\to 0} \frac{\Phi_t^*(f dx^j) - f dx^j}{t}$$
$$= \lim_{t\to 0} \frac{f \circ \Phi_t d(x^j \circ \Phi_t) - f dx^j}{t}.$$

If we denote Φ^j to be $x^j \circ (\varphi \circ \Phi \circ \varphi^{-1})$. Similarly, abuse of some other conventions. Because Φ is the phase flow of the field, we have

$$\frac{d\Phi^j}{dt} = X^j, \quad j = 1, 2, \cdots, n.$$

It implies that

$$\Phi_t^j(x^1, x^2, \cdots, x^n) = x^j + X^j t + o(t), \quad j = 1, 2, \cdots, n.$$

Hence

$$f \circ \Phi_t d(x^j \circ \Phi_t) = f \circ \Phi_t \sum_{i=1}^n \frac{\partial \Phi_t^j}{\partial x^i} dx^i$$
$$= f \circ \Phi_t \sum_{i=1}^n \left(\delta_i^j + t\frac{\partial X^j}{\partial x^i} + o(t) \right) dx^i.$$

Now it is easy to see that

$$\lim_{t\to 0}\frac{f\circ\Phi_t\sum_{i\neq j}^{n}\left(\delta_i^j + t\frac{\partial X^j}{\partial x^i} + o(t)\right)dx^i}{t} = f\sum_{i\neq j}^{n}\frac{\partial X^j}{\partial x^i}dx^i. \tag{4}$$

On the other hand, we have

$$\lim_{t\to 0}\frac{f(\Phi_t)-f}{t} = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\frac{d\Phi_t}{dt}\Big|_{t=0} = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}X^i$$

therefore

$$\lim_{t\to 0}\frac{f\circ\Phi_t\left(\delta_j^j + t\frac{\partial X^j}{\partial x^j} + o(t)\right)dx^j - f dx^j}{t} = \left(\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}X^i + f\frac{\partial X^j}{\partial x^j}\right)dx^j. \tag{5}$$

From (4) and (5), we conclude that (2) is true for $\omega = f dx^j$, hence for general one form.

The same method can also be used to show (3), we just need to notice that

$$\Phi_*\left(f\frac{\partial}{\partial x^j}\right) = f\circ\Phi_{-t}\Phi_*\left(\frac{\partial}{\partial x^j}\right) = f\circ\Phi_{-t}\sum_{i=1}^{n}\frac{\partial\Phi^i}{\partial x^j}\frac{\partial}{\partial x^i}.$$

That is why we have a minus $-$ in (3). ■

2. Let M^{2n-2}, X_{q^1} and X_{p_1} be the sub-manifold and vector fields introduced in the proof of *Darboux's Theorem*. Let the vectors $\tau_1, \cdots, \tau_{2n-2}$ form a basis in the tangent space $T_{x_0}(M^{2n-2})$. Prove that the vectors,

$$X_{q^1}(x_0), X_{p_1}(x_0), \tau_1, \cdots, \tau_{2n-2}$$

are linearly independent.

Proof. *Note that M^{2n-2} is the level manifold introduced by*

$$\begin{cases} q^1(x^1,\cdots,x^{2n}) = 0,\\ p_1(x^1,\cdots,x^{2n}) = 0,\end{cases}$$

for any $X_0 \in T_{x_0}(M^{2n-2})$, we have

$$\Omega(X_0, X_{q^1}(x_0)) = \langle X_0, dq^1\rangle = \sum_{j=1}^{2n} X_0^j\frac{\partial q^1}{\partial x^j} = 0$$

by the problem 2 of HW 1. Similarly, $\Omega(X_0, X_{p_1}(x_0)) = 0$.

By the construction of q^1 *and* p_1*, we have*

$$\Omega(X_{q^1}(x_0), X_{p_1}(x_0)) = \{p_1, q^1\}(x_0) = -1.$$

Now, if

$$\mathcal{X} = \alpha X_{q^1}(x_0) + \beta X_{p_1}(x_0) + \sum_{i=1}^{2n-2} k_i \tau_i = 0,$$

then

$$\alpha = \Omega(X_{p_1}(x_0), \mathcal{X}) = 0 \quad \textit{and} \quad \beta = \Omega(\mathcal{X}, X_{q^1}(x_0)) = 0.$$

Hence

$$\mathcal{X} = \sum_{i=1}^{2n-2} k_i \tau_i = 0,$$

it follows that $k_i \equiv 0$ *since* $\tau_1, \cdots, \tau_{2n-2}$ *form a basis in the tangent space* $T_{x_0}(M^{2n-2})$.

Above all, we conclude

$$X_{q^1}(x_0), X_{p_1}(x_0), \tau_1, \cdots, \tau_{2n-2}$$

are linearly independent. ■

3. Let $B_{ji}(F)$ be the matrix of the Poisson brackets,

$$\{F_i, \phi_j\} = B_{ji}(F),$$

introduced in the proof of the *Liouville Theorem*. Show, that the set of equations,

$$\sum_r \left(B_{jr} \frac{\partial B_{kl}}{\partial F_r} - B_{kr} \frac{\partial B_{jl}}{\partial F_r} \right) = 0, \quad \forall\, j, k, l, \tag{6}$$

is the compatibility of the system,

$$\frac{\partial s_k}{\partial F_i} = (B^{-1})_{i,k}.$$

Proof. *This suffices to show that* (6) *is equivalent to*

$$\frac{\partial B^{ik}}{\partial F_j} = \frac{\partial B^{jk}}{\partial F_i},$$

where $B^{ik} = (B^{-1})_{ik}$*, for any* i, j, k.

Actually,

$$\begin{aligned}
& \sum_r \left(B_{jr} \frac{\partial B_{kl}}{\partial F_r} - B_{kr} \frac{\partial B_{jl}}{\partial F_r} \right) = 0, \quad \forall\, j, k, l, \\
\iff \quad & \sum_r B_{jr} \frac{\partial B_{kl}}{\partial F_r} = \sum_r B_{kr} \frac{\partial B_{jl}}{\partial F_r}, \quad \forall\, j, k, l, \\
\iff \quad & \sum_{k,r} B^{ik} B_{jr} \frac{\partial B_{kl}}{\partial F_r} = \sum_{k,r} B^{ik} B_{kr} \frac{\partial B_{jl}}{\partial F_r}, \quad \forall\, j, l, i, \\
\iff \quad & \sum_{k,r} B^{ik} B_{jr} \frac{\partial B_{kl}}{\partial F_r} = \frac{\partial B_{jl}}{\partial F_i}, \quad \forall\, j, l, i. \qquad (7)
\end{aligned}$$

On the other hand, note that

$$\sum_k B^{ik} B_{kl} = \delta^i_l \Longrightarrow \frac{\partial B^{ik}}{\partial F_j} B_{kl} = -\sum_k B^{ik} \frac{\partial B_{kl}}{\partial F_j}.$$

Similarly,

$$\sum_k \frac{\partial B^{jk}}{\partial F_i} B_{kl} = -\sum_k B^{jk} \frac{\partial B_{kl}}{\partial F_i}.$$

Therefore,

$$\begin{aligned}
& \frac{\partial B^{ik}}{\partial F_j} = \frac{\partial B^{jk}}{\partial F_i}, \quad \forall\, i, j, k, \\
\iff \quad & \sum_k \frac{\partial B^{ik}}{\partial F_j} B_{kl} = \sum_k \frac{\partial B^{jk}}{\partial F_i} B_{kl}, \quad \forall\, i, j, l, \\
\iff \quad & \sum_k B^{ik} \frac{\partial B_{kl}}{\partial F_j} = \sum_k B^{jk} \frac{\partial B_{kl}}{\partial F_i}, \quad \forall\, i, j, l, \\
\iff \quad & \sum_{j,k} B_{rj} B^{ik} \frac{\partial B_{kl}}{\partial F_j} = \sum_{j,k} B_{rj} B^{jk} \frac{\partial B_{kl}}{\partial F_i}, \quad \forall\, i, r, l, \\
\iff \quad & \sum_{j,k} B_{rj} B^{ik} \frac{\partial B_{kl}}{\partial F_j} = \frac{\partial B_{rl}}{\partial F_i}, \quad \forall\, i, r, l. \qquad (8)
\end{aligned}$$

Exchange r and j in (7), we get (8), thus completing the proof. ■

4. Consider the equations of the *Euler Top*, i.e. the equations of rotation of a rigid body about its fixed center of mass,

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times I^{-1}\mathbf{M}.$$

Here $\mathbf{M} = (M_1, M_2, M_3)$ is the angular momentum of the body, I_1, I_2, I_3 are the principal moments of inertia, and $I^{-1}\mathbf{M} \equiv (I_1^{-1}M_1, I_2^{-1}M_2, I_3^{-1}M_3)$. Show that these equations can be interpreted as the Hamiltonian system with respect to the Kirillov symplectic form on the $\mathrm{SO}(3)$-orbits (see Bonus 2 of HW 1) and with the Hamiltonian,

$$H(M) = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3}.$$

Remark. *In my answer to HW 1, I get the Kirillov symplectic form to be*

$$\Omega = -2r\sin\phi d\phi \wedge d\theta.$$

Here I think I should use $\Omega = -r\sin\phi d\phi \wedge d\theta$.

Proof. *Let* $M_1 = r\sin\phi\cos\theta$, $M_2 = r\sin\phi\sin\theta$, $M_3 = r\cos\phi$, *where* $\phi \in [0,\pi)$ *and* $\theta \in [0, 2\pi)$, *then*

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times I^{-1}\mathbf{M}$$

can be transformed.

In fact,

$$\frac{dM_1}{dt} = M_2M_3\left(\frac{1}{I_3} - \frac{1}{I_2}\right)$$

is transformed to be

$$\cos\phi\cos\theta\frac{d\phi}{dt} - \sin\phi\sin\theta\frac{d\theta}{dt} = r\sin\phi\cos\phi\sin\theta\left(\frac{1}{I_3} - \frac{1}{I_2}\right). \tag{9}$$

Similarly,

$$\frac{dM_2}{dt} = M_3M_1\left(\frac{1}{I_1} - \frac{1}{I_3}\right)$$

is transformed to be

$$\cos\phi\sin\theta\frac{d\phi}{dt} + \sin\phi\cos\theta\frac{d\theta}{dt} = r\sin\phi\cos\phi\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_3}\right). \tag{10}$$

$$\frac{dM_3}{dt} = M_1M_2\left(\frac{1}{I_2} - \frac{1}{I_1}\right)$$

is transformed to be

$$-\sin\phi\frac{d\phi}{dt} = r\sin^2\phi\sin\theta\cos\theta\left(\frac{1}{I_2} - \frac{1}{I_1}\right). \tag{11}$$

From (9), (10) *and* (11), *it follows that*

$$\frac{d\phi}{dt} = r\sin\phi\sin\theta\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_2}\right) \tag{12}$$

and

$$\frac{d\theta}{dt} = r\cos\phi\left(\frac{\cos^2\theta}{I_1} + \frac{\sin^2\theta}{I_2} - \frac{1}{I_3}\right). \tag{13}$$

We just need to check

$$X_H = \frac{d\phi}{dt}\frac{\partial}{\partial\phi} + \frac{d\theta}{dt}\frac{\partial}{\partial\theta}.$$

Note that

$$H(M) = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3},$$

it follows that

$$H = \frac{r^2}{2}\left(\frac{\sin^2\phi\cos^2\theta}{I_1} + \frac{\sin^2\phi\sin^2\theta}{I_2} + \frac{\cos^2\phi}{I_3}\right).$$

Hence

$$\begin{aligned} dH &= \frac{\partial H}{\partial\phi}d\phi + \frac{\partial H}{\partial\theta}d\theta \\ &= r^2\sin\phi\cos\phi\left(\frac{\cos^2\theta}{I_1} + \frac{\sin^2\theta}{I_2} - \frac{1}{I_3}\right)d\phi \\ &\quad + r^2\sin^2\phi\sin\theta\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_2}\right)d\theta. \end{aligned}$$

If we denote X_H *by* $A\frac{\partial}{\partial\phi} + B\frac{\partial}{\partial\theta}$*, then*

$$\Omega\left(X_H, \frac{\partial}{\partial\phi}\right) = \left\langle \frac{\partial}{\partial\phi}, dH \right\rangle$$

deduces that (here I use $\Omega = -r\sin\phi d\phi\wedge d\theta$ *)*

$$Br\sin\phi = r^2\sin\phi\cos\phi\left(\frac{\cos^2\theta}{I_1} + \frac{\sin^2\theta}{I_2} - \frac{1}{I_3}\right),$$

thus

$$B = r\cos\phi\left(\frac{\cos^2\theta}{I_1} + \frac{\sin^2\theta}{I_2} - \frac{1}{I_3}\right) = \frac{d\theta}{dt}.$$

Similarly, from $\Omega\left(X_H, \frac{\partial}{\partial\theta}\right)$*, it follows that*

$$A = r\sin\phi\sin\theta\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_2}\right) = \frac{d\phi}{dt}.$$

Above all, the proof is complete. ■

5. Show that, in the spherical coordinates on the sphere,

$$M_1^2 + M_2^2 + M_3^2 = c,$$

the Euler equations are reduced to the single equation,

$$\sin\phi \frac{d\phi}{dt} = \pm c\sqrt{l_1 \cos^4\phi + l_2 \cos^2\phi + l_3} \tag{14}$$

with some constant parameters l_j which are the functions of the moments of inertia I_j and the energy,

$$E \equiv \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3}.$$

Remark. *Equation (14) is solvable in the Jacobi elliptic functions.*

Proof. *From (12) and (13), we have*

$$\frac{d\phi}{d\theta} = \frac{\beta \sin\phi \sin\theta \cos\theta}{\cos\phi(\alpha - \beta\sin^2\theta)}, \tag{15}$$

where

$$\alpha = \frac{1}{I_1} - \frac{1}{I_3} \quad \text{and} \quad \beta = \frac{1}{I_1} - \frac{1}{I_2}.$$

By (15), it follows that

$$\frac{2d\sin\phi}{\sin\phi} = \frac{d(\beta\sin^2\theta)}{\alpha - \beta\sin^2\theta}.$$

Threfore, there is some parameter p such that

$$\sin^2\phi(a - \beta\sin^2\theta) = p.$$

From this expression we can deduce that

$$\begin{aligned}\left(\sin\phi\frac{d\phi}{dt}\right)^2 &= r^2\beta\sin^2\phi\sin^2\theta\beta\sin^2\phi\cos^2\theta \\ &= r^2(\alpha - \alpha\cos^2\phi - p)(p - \alpha - \beta - (\alpha - \beta)\cos^2\phi) \\ &= r^2(l_1\cos^4\phi + l_2\cos^2\phi + l_3),\end{aligned}$$

where l_1, l_2 and l_3 are functions with respect to α, β and p. Because α and β are functions of I_j. While we can determine p from the energy and I_3, since

$$\begin{aligned}E &= \frac{r^2}{2}\left(\frac{\sin^2\phi\cos^2\theta}{I_1} + \frac{\sin^2\phi\sin^2\theta}{I_2} + \frac{\cos^2\phi}{I_3}\right) \\ &= \frac{r^2}{2}\left(\alpha\sin^2\phi - \beta\sin^2\phi\sin^2\theta + \frac{1}{I_3}\right) \\ &= \frac{r^2}{2}\left(p + \frac{1}{I_3}\right).\end{aligned}$$

Hence the Euler equations are reduced to the single equation,

$$\sin\phi\frac{d\phi}{dt} = \pm c\sqrt{l_1\cos^4\phi + l_2\cos^2\phi + l_3}$$

with some constant parameters l_j which are the functions of the moments of inertia I_j and the energy E, where $r = c$. ■