Homework Assignment ♯ 1

Due date: Wednesday, November 1

Remark. *Here I use the book below as reference. Download Digital Version*

[L] John M.Lee, *Introduction to Smooth Manifolds, Graduate Texts in Mathematics, **218** (2012), 2nd Edition.*

1. Let

$$\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{Mat}(n,\mathbb{R}) : AA^T = I\}.$$

Show that $\mathrm{O}(n,\mathbb{R})$ is a smooth manifold of dimension $\dfrac{n(n-1)}{2}$.

Proof. *Let $\mathrm{S}(n,\mathbb{R})$ denote the set of symmetric $n \times n$ matrices, which is easily seen to be a linear subspace of $\mathrm{Mat}(n,\mathbb{R})$ of dimension $n(n+1)/2$. Define $\varphi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{S}(n,\mathbb{R})$ by $\varphi(A) = AA^T$, then $\mathrm{O}(n,\mathbb{R}) = \varphi^{-1}(I)$ and φ is smooth since each components of φ is a polynomial of components of A. If we can prove that I is a regular value of φ, then by Corollary 5.14 Regular Level Set Theorem Of [L], we know that $\mathrm{O}(n,\mathbb{R})$ is a embedded submanifold of $\mathrm{GL}(n,\mathbb{R})$, and its dimension is $n^2 - n(n+1)/2 = n(n-1)/2$ ($\dim \mathrm{GL}(n,\mathbb{R}) - \mathrm{S}(n,\mathbb{R})$).*

Let $A \in \mathrm{O}(n,\mathbb{R})$, we will show that $\varphi_* : T_A\mathrm{GL}(n,\mathbb{R}) \to T_{\varphi(A)}\mathrm{S}(n,\mathbb{R})$ is surjective. For any $X \in T_A\mathrm{GL}(n,\mathbb{R}) \cong \mathrm{Mat}(n,\mathbb{R})$(which will be proved in Problem 4), assume $\gamma : (-\epsilon,\epsilon) \to \mathrm{GL}(n,\mathbb{R})$ be a smooth curve on $\mathrm{GL}(n,\mathbb{R})$ such that $\gamma(t) = A + tX$, which satisfies $\gamma(0) = A$ and $\gamma'(0) = X$. Then

$$\varphi_*(X) = \varphi_*(\gamma'(0)) = \frac{d}{dt}\Big|_{t=0}\varphi(\gamma(t)) = XA^T + AX^T. \tag{1}$$

For any $C \in \mathrm{S}(n,\mathbb{R})$, we have

$$\varphi_*(CA/2) = CAA^T/2 + AA^TC^T/2 = C/2 + C^T/2 = C,$$

i.e. $\varphi_*|_A$ is surjective. The proof is complete. ■

Remark. *Here, we use some results such as identifying tangent vectors to a finite-dimensional linear space with elements of the space itself (Proposition 3.13 of [L]), Regular Value and Submersion Theorem or Regular Level Set Theorem.*

2. Let $f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, N-n$ be collection of smooth functions such that
$$\operatorname{rank}\left\{\frac{\partial f_i}{\partial x_j}\right\} = N - n. \tag{2}$$
Define the smooth manifold M by the nonlinear equations,
$$M = \{\mathbf{x} : f_i(\mathbf{x}) = 0,\ i = 1, \cdots, N-n\}.$$
Show that the tangent space $T_{\mathbf{x}_0}$ to M at point $\mathbf{x}_0$ is defined by the linear equations,
$$T_{\mathbf{x}_0} = \{\mathbf{X} : \langle \mathbf{X}, \operatorname{grad} f_i(\mathbf{x}_0)\rangle = 0,\ i = 1, \cdots, N-n\}.$$

Remark. *First, we show that M is a manifold of dimension n. Since $\operatorname{rank}\left\{\frac{\partial f_i}{\partial x_j}\right\} = N-n$, we assume that the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{(N-n)\times(N-n)}$ is invertible at some given point $\mathbf{x}_0$ for convenience, where $1 \leqslant i, j \leqslant N-n$. Hence by Implicit Function Theorem, there exist neighbourhood $V_0 \subset \mathbb{R}^{N-n}$ of $(\mathbf{x}_1, \cdots, \mathbf{x}_{N-n}) = p$ and $W_0 \subset \mathbb{R}^n$ of $(\mathbf{x}_{N-n+1}, \cdots, \mathbf{x}_N) = q$ and a smooth map $\varphi : W_0 \to V_0$ such that*
$$f_i(x_1, \cdots, x_N) = 0,\ i = 1, \cdots, N-n$$
for any $(x_1, \cdots, x_N) \in V_0 \times W_0$ if and only if $\varphi(x_{N-n+1}, \cdots, x_N) = (x_1, \cdots, x_{N-n})$. Thus we can construct a diffeomorphism $\phi : (V_0 \times W_0) \cap M \to W_0$ send $(x_1, \cdots, x_N)$ to $(x_{N-n+1}, \cdots, x_N)$, which is a local coordinate map. Now, it is easy to see M is a manifold of dimension n.

Proof. *Of course, we can prove this is a manifolds in another way by some useful theorems in [L]. Define $\varphi : \mathbb{R}^N \to \mathbb{R}^{N-n}$ by*
$$\varphi(x) = (f_1(x), \cdots, f_{N-n}(x)),$$
then $M = \varphi^{-1}(0)$. Note that (2) means φ is a submersion, by Corollary 5.13 Submersion Level Set Theorem of [L], it follows that $M = \varphi^{-1}(0)$ is an embedded manifold with dimension n.

By Proposition 5.38 of [L], $T_{\mathbf{x_0}} = \ker d\varphi_{\mathbf{x_0}}$. For any $\mathbf{X} \in T_{\mathbf{x}_0}(\mathbb{R}^N)$, let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^N$ such that $\gamma(0) = \mathbf{x}_0$ and $\gamma'(0) = \mathbf{X}$. Then, one has
$$d\varphi_{\mathbf{x}_0}(\mathbf{X}) = \left(\frac{df_1(\gamma(t))}{dt}\Big|_{t=0}, \cdots, \frac{df_{N-n}(\gamma(t))}{dt}\Big|_{t=0}\right).$$

By Chain Rule, it follows that

$$\frac{df_i(\gamma(t))}{dt}\Big|_{t=0} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}\Big|_{\mathbf{x}_0} \frac{dx_i}{dt}\Big|_{t=0} = \langle \mathbf{X}, \mathrm{grad}\, f_i(\mathbf{x}_0)\rangle.$$

That means $T_{\mathbf{x}_0} = \ker d\varphi_{\mathbf{x_0}} = \{\mathbf{X} : \langle \mathbf{X}, \mathrm{grad}\, f_i(\mathbf{x}_0)\rangle = 0,\ i = 1, \cdots, N-n\}$. ■

Remark. *In fact, what I used it is equivalent to the problem. And here we actually identify the tangent space as a subspace of* $T_{\mathbf{x}_0}(\mathbb{R}^N)$*, so all what we need is the Chain Rule.*

3. Let $X \in T_p(M)$ be a tangent vector defined as a linear functional on $F(M)$ satisfying the Leibniz Rule,

$$X(fg) = X(f)g(p) + X(g)f(p). \tag{3}$$

Show that $X(\mathrm{const}) = 0$.

Proof. *Since* X *is linear, it suffices to show that* $X(1) = 0$*, where* 1 *means the constant map (hence smooth) sending* M *to* 1*. By (3), we obtain that*

$$X(1 \cdot 1) = X(1) \cdot 1 + X(1) \cdot 1 \quad \Longrightarrow \quad X(1) = 2X(1).$$

Hence $X(1) = 0$. ■

4. Prove the following equations concerning the tangent spaces of the "matrix" manifolds.

- $T_I(\mathrm{GL}(n,\mathbb{R})) = \mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R})$.
- $T_I(\mathrm{SL}(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R}) \equiv \{A \in \mathrm{Mat}(n,\mathbb{R}) : \mathrm{tr}\, A = 0\}$.
- $T_I(\mathrm{O}(n,\mathbb{R})) = \mathfrak{so}(n,\mathbb{R}) \equiv \{A \in \mathrm{Mat}(n,\mathbb{R}) : A = -A^T\}$.
- $T_B(\mathrm{SL}(n\mathbb{R})) = B\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{sl}(n,\mathbb{R})B$.

Proof. *As above (Remark of Problem 1), we always identify tangent vectors to a finite-dimensional linear space with elements of the space itself. Hence*

$$T_I(\mathrm{Mat}(n,\mathbb{R})) = \mathrm{Mat}(n,\mathbb{R}).$$

Since $\det : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ *is continuous and smooth, it follows that* $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ *is open in* $\mathrm{Mat}(n,\mathbb{R})$. *From the local property (Proposition 3.9 of [L]) of tangent vectors, it follows that*

$$\mathfrak{gl}(n,\mathbb{R}) = T_I(\mathrm{GL}(n,\mathbb{R})) = T_I(\mathrm{Mat}(n,\mathbb{R})) = \mathrm{Mat}(n,\mathbb{R}).$$

For any $X \in T_I(\mathrm{GL}(n,\mathbb{R}))$, *there exists a smooth curve* $\gamma : (-\epsilon,\epsilon) \to \mathrm{GL}(n,\mathbb{R})$ *such that* $\gamma(t) = I + tX$ *satisfies* $\gamma(0) = I$ *and* $\gamma'(0) = X$. *It follows that*

$$\det_*(X) = \frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \det(I + tX) = \mathrm{tr}\ X. \tag{4}$$

Note that (4) *also means that*

$$\det_* : T_I\mathrm{GL}(n,\mathbb{R}) \to R$$

is surjective, which also means $\det$ *is a submersion. Hence* $T_I(\mathrm{SL}(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R}) = \mathrm{Ker}\ \det_* = \{A \in \mathrm{Mat}(n,\mathbb{R}) : \mathrm{tr}\ A = 0\}$ *by Proposition 5.38 of [L].*

Similarly, from the process of Problem 1 and (1), *one has*

$$T_I(\mathrm{O}(n,\mathbb{R})) = \mathfrak{so}(n,\mathbb{R}) = \mathrm{Ker}\ \varphi_* = \{A \in \mathrm{Mat}(n,\mathbb{R}) : A = -A^T\}.$$

For any $X \in T_B(\mathrm{GL}(n,\mathbb{R}))$, *there exists a smooth curve* $\gamma : (-\epsilon,\epsilon) \to \mathrm{GL}(n,\mathbb{R})$ *such that* $\gamma(t) = B + tX$ *satisfies* $\gamma(0) = B$ *and* $\gamma'(0) = X$. *It follows that*

$$\begin{aligned}
\det_*(X) &= \frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \det(B + tX) \\
&= \frac{d}{dt}\Big|_{t=0} \det B \cdot \det(I + tB^{-1}X) = \det B \cdot \frac{d}{dt}\Big|_{t=0} \det(I + tB^{-1}X) \\
&= \mathrm{tr}\ (B^{-1}X).
\end{aligned}$$

By Proposition 5.38 of [L], this means $B^{-1}X \in \mathfrak{sl}(n,\mathbb{R})$ *if and only if* $X \in T_B(\mathrm{SL}(n,\mathbb{R}))$, *i.e.* $T_B(\mathrm{SL}(n,\mathbb{R})) = B\mathfrak{sl}(n,\mathbb{R})$. *In fact,* $T_B(\mathrm{SL}(n,\mathbb{R})) = B\mathfrak{sl}(n,\mathbb{R})$ *can be done by easy dimensional argument. Similarly, using* $\det(B + tX) = \det(I + XB^{-1}) \det B$ *will deduce* $T_B(\mathrm{SL}(n,\mathbb{R})) = \mathfrak{sl}(n,\mathbb{R})B$. ■

5. Let $\wedge$ be $n \times n$ diagonal matrix whose diagonal entries are $\lambda_1, \cdots, \lambda_n$ and they are all different, i.e. $\lambda_i \neq \lambda_k$, $i \neq k$. Define

$$O_\wedge = \{g \wedge g^{-1},\ g \in \mathrm{GL}(n,\mathbb{R})\}.$$

Prove that $O_\wedge$ is a smooth manifold of dimension $n(n-1)$ and

$$T_A(O_\wedge) = \{[A, X], \quad X \in \mathrm{Mat}(n,\mathbb{R})\}.$$

Proof. *Note that there is a natural left smooth action of* $\mathrm{GL}(n,\mathbb{R})$ *on* $\mathrm{Mat}(n,\mathbb{R})$*, which is defined by*

$$\Phi : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R}) \quad \Phi(g,A) = gAg^{-1}.$$

Then, the isotropy group (stabilizer of $\wedge$*)* $G_\wedge = \Phi^{-1}(\wedge)$ *is closed and* $\mathrm{GL}(n,\mathbb{R})$ *acts transitively on* $O_\wedge$*. Then by Theorem 21.20 of [L],* $O_\wedge$ *has an unique smooth manifold structure with respect to which the given action is smooth. Furthermore, with this structure,* $\dim O_\wedge = \dim \mathrm{GL}(n,\mathbb{R}) - \dim G_\wedge$*. From the condition of* $\wedge$*,* $g \in G_\wedge$ *if and only if* g *is diagonal and invertible. Consequently, it is easy to see* $G_\wedge$ *is of dimension* n*, which means* $\dim O_\wedge = n(n-1)$*.*

Define φ *by*

$$\varphi : \mathrm{GL}(n,\mathbb{R}) \to O_\wedge, \quad \varphi(g) = g \wedge g^{-1}.$$

Then φ *is a submersion by Theorem 4.14 Global Rank Theorem of [L], since* φ *is surjective. Now, we will determine* $T_A(O_\wedge)$*. Since* $A \in O_\wedge$*, we can assume* $A = g \wedge g^{-1}$ *for some* $g \in \mathrm{GL}(n,\mathbb{R})$*. Note that* φ_* *is surjective, then* $T_A(O_\wedge) = \varphi_*(T_g\mathrm{GL}(n,\mathbb{R}))$*. For any* $X \in T_g\mathrm{GL}(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R})$*, there exists a smooth curve* $\gamma : (-\epsilon,\epsilon) \to \mathrm{GL}(n,\mathbb{R})$ *such that* $\gamma(t) = g + tX$ *satisfying* $\gamma(0) = g$ *and* $\gamma'(0) = X$*. Then*

$$\varphi_*(X) = \varphi_*(\gamma'(0)) = \frac{d}{dt}\Big|_{t=0}\varphi(\gamma(t)) = \frac{d}{dt}\Big|_{t=0}(g+tX) \wedge (g+tX)^{-1}. \tag{5}$$

Denote $\mu(t) = (g+tX) \wedge (g+tX)^{-1}$*, then* $(g+tX)\wedge = \mu(t)(g+tX)$*. It follows that*

$$\frac{d}{dt}\Big|_{t=0}(g+tX)\wedge = \frac{d}{dt}\Big|_{t=0}(\mu(t)(g+tX)) = \frac{d}{dt}\Big|_{t=0}\mu(t)g + \mu(0)X,$$

i.e.

$$X\wedge = g\frac{d}{dt}\Big|_{t=0}\mu(t) + g \wedge g^{-1}X \Longrightarrow \frac{d}{dt}\Big|_{t=0}\mu(t) = [Xg^{-1}, g \wedge g^{-1}].$$

Note that $A = g \wedge g^{-1}$ *and* $\mathrm{Mat}(n,\mathbb{R})g^{-1} = \mathrm{Mat}(n,\mathbb{R})$*, we deduce that*

$$T_A(O_\wedge) = \{[A,X], \quad X \in \mathrm{Mat}(n,\mathbb{R})\},$$

thus completing the proof. ■

Remark. *Of course, we have a smooth map* $f : \mathrm{GL}(n,\mathbb{R}) \to M$ *defined by* $f(g) = g \wedge g^{-1}$*. Moreover,* $G_\wedge = f^{-1}(\wedge)$ *is a closed subgroup of* $\mathrm{GL}(n,\mathbb{R})$*. Hence, by Theorem 21.17 Homogeneous Space Construction Theorem of [L],* $\mathrm{GL}(n,\mathbb{R})/G_\wedge$ *is* *a smooth manifold with dimension* $\dim \mathrm{GL}(n,\mathbb{R}) - \dim G_\wedge$ *and the quotient map* $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/G_\wedge$ *is a submersion. Let* $G = \mathrm{GL}(n,\mathbb{R})$ *and* $H = G_\wedge$*, we can construct a commutative diagram below. Moreover, by Theorem 4.29 Characteristic Property of Surjective Smooth Submersions of [L], it follows that* $\tilde{f}$ *is smooth.*

$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ G/H \arrow[r, "\tilde{f}"'] & M \end{tikzcd}$$

It is easy to see $\tilde{f}$ *is an equivariant map and injective, then* $\tilde{f}$ *is a immersion. Hence deduce that the image of* $\tilde{f}$ *is smooth manifold, i.e.,* $O_\wedge$ *is a manifold. This is just a restatement of Theorem 21.20 of [L].*

6. Show that
$$\omega^1 \wedge \omega^2 = (-1)^{\deg \omega^1 \deg \omega^2} \omega^2 \wedge \omega^1.$$

Proof. *Denote* $\deg \omega^1$ *and* $\deg \omega^2$ *by* m_1 *and* m_2*, respectively. Define a permutation* σ *in symmetric group* $S_{m_1+m_2}$ *such that*

$$\sigma(1) = m_2 + 1, \sigma(2) = m_2 + 2, \cdots, \sigma(m_1) = m_2 + m_1,$$

$$\sigma(m_1 + 1) = 1, \sigma(m_1 + 2) = 2, \cdots, \sigma(m_1 + m_2) = m_2.$$

The sign of σ *is equal to* $(-1)^{m_1 m_2}$ *by counting the inversion number. Now for any smooth vector field* $X_1, \cdots, X_{m_1+m_2}$*, we have*

$$\begin{aligned}
& m_1! m_2! (\omega^1 \wedge \omega^2)(X_1, \cdots, X_{m_1+m_2}) \\
= & \sum_{\tau\sigma \in S_{m_1+m_2}} (\mathrm{sign}\ \tau\sigma) \omega^1(X_{\tau\sigma(1)}, \cdots, X_{\tau\sigma(m_1)}) \omega^2(X_{\tau\sigma(m_1+1)}, \cdots, X_{\tau\sigma(m_1+m_2)}) \\
= & \sum_{\tau \in S_{m_1+m_2}} (\mathrm{sign}\ \tau)(\mathrm{sign}\ \sigma) \omega^1(X_{\tau(m_2+1)}, \cdots, X_{\tau(m_1+m_2)}) \omega^2(X_{\tau(1)}, \cdots, X_{\tau(m_2)}) \\
= & (\mathrm{sign}\ \sigma) \sum_{\tau \in S_{m_1+m_2}} (\mathrm{sign}\ \tau) \omega^2(X_{\tau(1)}, \cdots, X_{\tau(m_2)}) \omega^1(X_{\tau(m_2+1)}, \cdots, X_{\tau(m_1+m_2)}) \\
= & (\mathrm{sign}\ \sigma) m_1! m_2! (\omega^2 \wedge \omega^1)(X_1, \cdots, X_{m_1+m_2}),
\end{aligned}$$

i.e. $\omega^1 \wedge \omega^2 = (-1)^{\deg \omega^1 \deg \omega^2} \omega^2 \wedge \omega^1$*, since* $X_1, \cdots, X_{m_1+m_2}$ *are arbitrary.* ■

7. Let $F : M \to N$ be a smooth map between two smooth manifolds. Show that the operations F^* and d commute, i.e.

$$F^*(d\omega) = dF^*(\omega), \quad \forall \omega \in \wedge^k(N). \tag{6}$$

Proof. *Here we acknowledge that d has local property, i.e. d is determined by local condition rather than global ones (of course this is true by the definition in class). Note that d and F^* are linear, it suffices to show (6) for $f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, where $f \in C^\infty(M)$ and $\{x^i\}$ is a local smooth coordinate system.*

We have

$$\begin{aligned}
& F^*(d(f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k})) \\
= \; & F^*(df \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}) \\
= \; & F^*(df) \wedge F^*(dx^{i_1}) \wedge F^*(dx^{i_2}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
= \; & d(f \circ F) \wedge F^*(dx^{i_1}) \wedge F^*(dx^{i_2}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
= \; & d((f \circ F)F^*(dx^{i_1}) \wedge F^*(dx^{i_2}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
= \; & d(F^*(f)F^*(dx^{i_1}) \wedge F^*(dx^{i_2}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
= \; & d(F^*(f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k})),
\end{aligned}$$

here we use the facts that $F^*(\mu \wedge \nu) = F^*\mu \wedge F^*\nu$, the associativity of wedge product and $F^* f = f \circ F$ for any $f \in C^\infty(N)$. ■

Remark. *Actually, we can also do it globally. Since we have the invariant formula for exterior derivatives, i.e.*

$$\begin{aligned}
d\omega(X_1, \cdots, X_{k+1}) = & \sum_{1 \leqslant i \leqslant k+1} (-1)^{i-1} X_i\big(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1})\big) \\
& + \sum_{1 \leqslant i < j \leqslant k+1} \omega\big([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}\big),
\end{aligned}$$

we can argue it by the similar facts used above.

8. Let $T^*(M)$ be the co-tangent bundle of a smooth manifold M. Define the linear differential form $\theta \in \wedge^1\big(T^*(M)\big)$ by the equations,

$$\langle X, \theta(x) \rangle = \langle \pi_*(X), x \rangle, \quad \forall\, x \in T^*(M), \text{ and } \forall\, X \in T_x\big(T^*(M)\big), \tag{7}$$

where π is a natural projection,

$$\pi : T^*(M) \to M.$$

Let $(q_1, \cdots, q_n)$ be the local coordinates on M and $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ be the induced local coordinates on $T^*(M)$. Show that,

$$\theta = \sum_{j=1}^{n} p_j dq^j.$$

Proof. *For any $x \in T^*(M)$, choose a local coordinate (φ, U) of $\pi(x)$ on M. Let $\varphi(\pi(x)) = (q^1, \cdots, q^n)$, if under the local coordinate system, we have*

$$x = \sum_{j=1}^{n} p_j dq^j\big|_{\pi(x)}.$$

Then, the induced local coordinate system $(\hat{\varphi}, U \times \mathbb{R}^n)$ of $T^*(M)$ that maps x to $(q^1, \cdots, q^n, p_1, \cdots, p_n)$. Now, we use $(y^1, \cdots, y^n)$ and $(x^1, \cdots, x^{2n})$ to express local system (φ, U) and $(\hat{\varphi}, U \times \mathbb{R}^n)$, respectively.

Note that if $\hat{\varphi}(x) = (x^1, \cdots, x^{2n})$, then

$$\varphi \circ \pi \circ \hat{\varphi}^{-1}(x^1, \cdots, x^{2n}) = (y^1, \cdots, y^n)$$

and

$$x = \sum_{j=1}^{n} x^{n+j} dx^j\big|_{\pi(x)} \tag{8}$$

where $y^i = x^i$ for any $i = 1, \cdots, n$. Hence

$$\pi_*\left(\frac{\partial}{\partial x^j}\Big|_x\right) = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}\Big|_{\pi(x)} = \begin{cases} \dfrac{\partial}{\partial y^j}\Big|_{\pi(x)}, & 1 \leqslant j \leqslant n; \\ 0, & n+1 \leqslant j \leqslant 2n. \end{cases} \tag{9}$$

Of course, we have

$$\theta(x) = \sum_{i=1}^{2n} \left\langle \frac{\partial}{\partial x^i}\Big|_x, \theta(x) \right\rangle dx^i\big|_x. \tag{10}$$

From (7), (9) and (8), it follows that

$$\left\langle \frac{\partial}{\partial x^i}\Big|_x, \theta(x) \right\rangle = \left\langle \pi_*\left(\frac{\partial}{\partial x^i}\Big|_x\right), x \right\rangle = \begin{cases} x^{n+i}, & 1 \leqslant i \leqslant n; \\ 0, & n+1 \leqslant i \leqslant 2n. \end{cases} \tag{11}$$

Combining (10), (11) and $(x^1, \cdots, x^{2n}) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$, we deduce that

$$\theta = \sum_{j=1}^{n} p_j dq^j.$$

Here we use $(x^1, \cdots, x^{2n})$ to substitute $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ in order to avoid ambiguity. ■

9. Let Ω be the Kirillov form defined on the $\mathrm{SO}(3)$-orbits,

$$O_{C_0} = \{gC_0g^{-1},\ g \in \mathrm{SO}(3),\ C_0 \subset \mathfrak{so}(3)\} \subset \mathfrak{so}(3).$$

That is,

$$\Omega(C)(X_1, X_2) = \mathrm{tr}\ (C[A_1, A_2]), \quad C \in O_{C_0},$$

where $X_j \equiv [A_j, C] \in T_C\,(O_{C_0})$.

Show that the form Ω admits the following representation in the spherical coordinates (ϕ, θ),

$$\Omega = f(r) \sin\phi d\phi \wedge d\theta,$$

where $f(r)$ is a function of the orbit's invariant,

$$r = \left(-\frac{1}{2}\mathrm{tr}\ C_0^2\right)^{1/2}.$$

Find $f(r)$.

Solution. *From the class, we know that if we identify* $X = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix} \in \mathfrak{so}(3)$ *as* $\mathbf{X} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$*, then for any* $X, Y \in \mathfrak{so}(3)$*, we have*

$$\mathrm{tr}\ (XY) = -2\mathbf{X} \cdot \mathbf{Y} \quad \textit{and} \quad [X, Y] = -\mathbf{X} \times \mathbf{Y}.$$

Then

$$\Omega(C)(X_1, X_2) = \mathrm{tr}\ (C[A_1, A_2]) = 2\mathbf{C} \cdot (\mathbf{A}_1 \times \mathbf{A}_2).$$

We have already calculated $\tilde{\Omega}(C)(X_1, X_2) = \frac{1}{2}\mathrm{tr}\,(C[X_1, X_2])$ *in class. Here, we restate the arguments. One has*

$$\begin{aligned} & \tilde{\Omega}(C)(X_1, X_2) = \mathbf{C} \cdot (\mathbf{X}_1 \times \mathbf{X}_2) \\ = \ & c_1(X_1^2X_2^3 - X_1^3X_2^2) + c_2(X_1^3X_2^1 - X_1^1X_2^3) + c_3(X_1^1X_2^2 - X_1^2X_2^1) \\ = \ & c_1dx^2 \wedge dx^3(X_1, X_2) + c_2dx^3 \wedge dx^1(X_1, X_2) + c_3dx^1 \wedge dx^2(X_1, X_2). \end{aligned}$$

It follows that $\tilde{\Omega} = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$.

For any $C \in O_{C_0}$, *one has*

$$-2\|\mathbf{C}\|^2 = \mathrm{tr} C^2 = \mathrm{tr}(gC_0g^{-1})^2 = \mathrm{tr} C_0^2 = -2\|\mathbf{C}_0\|^2 = -2r^2.$$

Hence, we can deduce that

$$O_{C_0} \subset \{\mathbf{C} \in \mathbb{R}^3 : \|\mathbf{C}\| = r\}.$$

Let $x^1 = r\sin\phi\cos\theta$, $x_2 = r\sin\phi\sin\theta$ *and* $x_3 = r\cos\phi$, *then we have*

$$\tilde{\Omega} = r^3 \sin\phi d\theta \wedge d\phi.$$

Since $\tilde{\Omega}(C)(X_1, X_2) = \mathbf{C} \cdot (\mathbf{X}_1 \times \mathbf{X}_2)$ *and*

$$\mathbf{C} \cdot (\mathbf{X}_1 \times \mathbf{X}_2) = \mathbf{C} \cdot [(\mathbf{A}_1 \times \mathbf{C}) \times (\mathbf{A}_2 \times \mathbf{C})] = \|\mathbf{C}\|^2 \mathbf{C} \cdot (\mathbf{A}_1 \times \mathbf{A}_2),$$

it follows that

$$\Omega = 2\tilde{\Omega} \div \|\mathbf{C}\|^2 = 2r\sin\phi d\theta \wedge d\phi = -2r\sin\phi d\phi \wedge d\theta,$$

i.e., $f(r) = -2r$. ■

Historic Remark about Darboux Theorem

The Darboux theorem was first proved (in a slightly different form) by Gaston Darboux in 1882, in connection with his work on ordinary differential equations arising in classical mechanics. The proof in [L] was discovered in 1971 by Alan Weinstein, based on a technique due to Jurgen Moser.

Weinstein's proof of the Darboux theorem is based on the theory of timedependent flows (see Theorem 9.48 of [L]). Page 571 of [L]